Smooth solution to the one-dimensional inhomogeneous non-automorphic Landau–Lifshitz equation

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Using the differential–difference method and viscosity vanishing approach, we obtain the existence and uniqueness of the global smooth solution to the periodic initial-value problem of the inhomogeneous, non-automorphic Landau–Lifshitz equation without Gilbert damping terms in one dimension. To establish the uniform estimates, we use some identities resulting from the fact \(|\mathbf{Z}(x, t)| = 1\) and the fact that the vectors \(\{\mathbf{Z}, \mathbf{Z}_x, \mathbf{Z} \times \mathbf{Z}_x\}\) form an orthogonal base of the space \(\mathbb{R}^3\).

Keywords: Landau–Lifshitz equation; inhomogeneous non-automorphic Heisenberg chain equation; existence; uniqueness; smooth solution

1. Introduction

In this paper, we are concerned with the existence and uniqueness of the smooth solution to the one-dimensional periodic initial-value problem of the inhomogeneous non-automorphic Landau–Lifshitz equation,

\[
\mathbf{Z}_t = f(x, t)\mathbf{Z} \times \mathbf{Z}_{xx} + \frac{\partial f(x, t)}{\partial x} \mathbf{Z} \times \mathbf{Z}_x, \tag{1.1}
\]

\[
\mathbf{Z}(x, 0) = \mathbf{Z}_0(x), \quad \mathbf{Z}(x + D, t) = \mathbf{Z}(x - D, t), \quad |\mathbf{Z}_0(x)| \equiv 1, \quad x \in \mathbb{R}^1, \tag{1.2}
\]

where \(D>0\) is a constant, \(f(x, t)\) and \(\mathbf{Z}_0(x)\) are smooth functions and \(f(x, t) \geq f_0 > 0\) for some constant \(f_0\), \(\mathbf{Z}(x, t) = (Z_1(x, t), Z_2(x, t), Z_3(x, t))\). We assume that \(f(x, t)\) and \(\mathbf{Z}_0(x)\) are periodic functions with period \(2D\).

Equation (1.1) is related to the generalized model of inhomogeneous ferromagnetisms and the simplified compressible ferromagnetisms.

As we know, the equation for the spin wave of inhomogeneous ferromagnetic medium was raised in the condensed matter physics given by Balakrishnan (1982). It can be derived from the inhomogeneous, isotropic Heisenberg exchange Hamiltonian

\[
\mathcal{H} = -J \sum_{i=1}^{N-1} f_i \mathbf{Z}_i : \mathbf{Z}_{i+1}
\]

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for a magnetic chain with a site-dependent nearest-neighbour interaction. The equation of motion obtained for $Z_i$ is

$$\frac{dZ_i}{dt} = Jf_i(Z_i \times Z_{i+1}) + Jf_{i-1}(Z_i \times Z_{i-1}),$$

where $J$ is the exchange constant. This equation can be shown to be valid in both the quantum and the classical cases. A continuum description in which $Z_i \to Z(x, t)$, $f_i \to f(x)$ is suitable when $Z_i, f_i$ vary slowly over one lattice separation $a$. Inserting Taylor expansions of $Z(x + a, t)$ and $f(x - a, t)$ in the above equation, we get in continuum limit

$$Z_t = f(x)Z \times Z_{xx} + f'(x)Z \times Z_x,$$  \tag{1.3}

where $Z(x, t) = (Z_1(x, t), Z_2(x, t), Z_3(x, t))$. Then, equation (1.1) is generalized from (1.3) by replacing $f(x)$ by $f(x, t)$, that is, the inhomogeneous function depends on time. Then, we call (1.1) the non-automorphic equation.

Another background of equation (1.1) is the model of compressible ferromagnetisms. Fievez (1982) revisited the one-dimensional classical compressible Heisenberg chain described by the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} P_i^2/2m + \frac{\alpha}{2} \sum_{i=1}^{N-1} (x_{i+1} - x_i)^2 - J \sum_{i=1}^{N-1} Z_i \cdot Z_{i+1} - \varepsilon \sum_{i=1}^{N-1} (x_{i+1} - x_i)Z_i \cdot Z_{i+1},$$

considered earlier by Cieplak & Turski (1980a,b), where $x_i$ is the displacement of the magnetic ion from equilibrium, without spin-phonon coupling, $\alpha$ is the spring constant and $\varepsilon = Jx$. In the continuum limit, which corresponds to long-wavelength excitations, the equations of motion deduced by Fievez read as

$$m\ddot{\eta} = \alpha \dot{\eta} + \varepsilon \frac{\partial}{\partial x} (Z')^2,$$  \tag{1.4}

$$\dot{Z} = \frac{\partial}{\partial x} \left\{ (J + \varepsilon \eta')Z \times Z' \right\},$$  \tag{1.5}

where $Z(x, t) = (Z_1(x, t), Z_2(x, t), Z_3(x, t)) \in \mathbb{R}^3$ and the substitution $x_i \to \eta(x, t)$, $Z_i \to Z(x, t)$ has been made, a dot denotes derivative with respect to $t$, a prime with respect to $x$.

Fievez tried the solution of the form $Z = \{ \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta \}$, $\eta = ax + f(u)$, where $\theta(x, t) = \varphi(u)$, $\varphi(x, t) = \varphi(t) + \Omega t$ with $u = x - ct$ (the lattice and spin wave are assumed to travel at the same velocity $c$). Equation (1.4) now becomes

$$(mc^2 - \alpha)f'' = \frac{\varepsilon}{2} \frac{\partial}{\partial x} (Z')^2,$$

with boundary conditions $f'(-\infty) = 0$, $Z'(-\infty) = 0$, $Z(-\infty) = 1$.

By integration, one has $(mc^2 - \alpha)f' = (\varepsilon/2)(Z')^2$ and $(Z')^2 = \theta^2 + \sin^2 \theta \varphi^2$.

Hence, Fievez derived the following Heisenberg chain equation:

$$Z_t = \{ G(Z_x)Z \times Z_x \}_x,$$  \tag{1.6}

in which $G(\xi) = A + B|\xi|^2$ with $A = J + \varepsilon \alpha$, $B = (\varepsilon^2/2)(mc^2 - \alpha)$, where $mc^2 > \alpha$. 

The solitons of (1.6) were given by Magyari (1982). Equation (1.6) is called the compressible Heisenberg chain equation (or compressible Landau–Lifshitz equation). Equation (1.6) with $B=0$, $A=f(x)$ corresponds to the inhomogeneous Heisenberg chain equation derived by Balakrishnan (1982), and (1.6) with $B=0$, $A=f(x, t)$ is just equation (1.1).

To our knowledge, there are some discussions on the periodic initial-value problems of inhomogeneous and compressible equation. Ding et al. (1999a) obtained the existence of measure-valued solution to the compressible equation (1.6). In the same year, these authors obtained the existence and uniqueness of the smooth solution to the inhomogeneous equation (1.3) by the viscosity vanishing method.

The novelties of this paper are in the following aspects. First of all, we find some new energy laws, which are different from those in the article given by Ding et al. (1999b) to establish the a priori estimates. Secondly, the most interesting one is to apply the property that the family of vectors $\{Z, Z_x, Z \times Z_x\}$ forms an orthogonal base of $\mathbb{R}^3$ since $|Z|=1$. This fact was first introduced by Zhou et al. (1991) for the case $f(x, t) \equiv 1$. Some other identities resulting from the fact that $|Z|=1$ are also used in the estimates. For the inhomogeneous model given by Ding et al. (1999b), they did not successfully use these ideas. Finally, since $f(x, t)$ depends on time $t$, we confront some special difficulties in the estimates. However, we have given sufficient conditions to get the existence of the smooth solutions to problems (1.1) and (1.2), in the case where the inhomogeneous function depends on time.

In order to prove the existence of smooth solutions to problems (1.1) and (1.2), we use the viscosity vanishing method. First, by the difference method, we establish the existence of smooth solutions to problems (1.1) and (1.2). In this procedure, the usage of the orthogonal base $\{Z, Z_x, Z \times Z_x\}$ may simplify our proof and make the idea clear compared with the proof in the paper given by Ding et al. (1999b). Some other orthogonal bases are also used in the proof. Again, since $|Z|=1$, we have used many other useful identities as we can see in §3.

As pointed out by Zhou et al. (1991) and Ding et al. (1999b), in the classical sense, equation (1.7) is equivalent to

$$Z_t = \varepsilon Z_{xx} + \varepsilon |Z_x|^2 Z + f(x, t)Z \times Z_{xx} + \frac{\partial f(x, t)}{\partial x} Z \times Z_x, \quad \varepsilon > 0,$$

(1.7)

$$Z(x, 0) = Z_0(x), \quad Z(x + D, t) = Z(x - D, t), \quad |Z_0(x)| \equiv 1, \quad x \in \mathbb{R}^1. \quad (1.8)$$

Then we prove some a priori estimates for the solutions of (1.7) and (1.8) uniform in $\varepsilon$ by constructing some new energy laws, we then send $\varepsilon$ to zero to obtain the existence of smooth solutions to problems (1.1) and (1.2). In this procedure, the usage of the orthogonal base $\{Z, Z_x, Z \times Z_x\}$ may simplify our proof and make the idea clear compared with the proof in the paper given by Ding et al. (1999b). Some other orthogonal bases are also used in the proof. Again, since $|Z|=1$, we have used many other useful identities as we can see in §3.

As pointed out by Zhou et al. (1991) and Ding et al. (1999b), in the classical sense, equation (1.7) is equivalent to

$$Z_t = -\varepsilon Z \times (Z \times Z_{xx}) + f(x, t)Z \times Z_{xx} + \frac{\partial f(x, t)}{\partial x} Z \times Z_x, \quad \varepsilon > 0,$$

(1.9)

since $A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$ and $|Z(x, t)| \equiv 1$.

Here, and in the following, when discussing the periodic problem, we assume that $f(x, t)$ and $Z_0(x)$ are periodic functions. When discussing the initial-value problem, we first replace them by $f^D(x, t)$ and $Z_0^D(x)$, where $f^D(x, t) = f(x, t)$ and $Z_0^D(x) = Z_0(x)$ in $-D \leq x \leq D$ and periodic in $\mathbb{R}$ and then let $D \to \infty$. Denote $\Omega = (-D, D), Q = \Omega \times [0, +\infty)$. 

2. $\epsilon > 0$: local smooth solution

To get the existence of local smooth solutions of (1.7) and (1.8), we apply the difference method. The proof is generally similar to that of the paper given by Ding et al. (1999b), since we only make a difference in the space direction. However, for completion, we give the proof in this section. For simplicity, we let $\epsilon = 1$.

We need the following well-known lemmas.

**Lemma 2.1 (Zhou et al. (1991)).** Let $q$, $r$ be real numbers and $j$, $m$ be integers such that $1 \leq q, r \leq \infty$, $0 \leq j < m$. If $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$, then

$$
\|D^j u\|_p \leq C \|u\|_q^{1-\alpha} \|D^m u\|_r^{\alpha},
$$

where $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, $p \geq 1$, $(j/m) \leq \alpha \leq 1$ and

$$
\frac{1}{p} - j = \frac{1-\alpha}{q} + \alpha \left(\frac{1}{r} - m\right), \quad \Omega \subset \mathbb{R}^1.
$$

**Lemma 2.2 (Zhou (1984)).** Let $p$ be a real number and $j$, $m$ be integers such that $2 \leq p \leq \infty$, $0 \leq j < m$. Then

$$
\|\delta^j u_h\|_p \leq C \|u_h\|_2 \left(\|\delta^m u_h\|_2 + \frac{\|u_h\|_2}{(2D)^m}\right)^\alpha,
$$

where $u_h = \{u_j = u(x_j)\mid j = 0, 1, 2, \ldots, J\}$, $x_j = jh$, $h = 2D/J$, $\alpha = (1/m)(j + (1/2) - (1/p))$,

$$
\|\delta^k u_h\|_p = \left(\sum_{i=0}^{J-k} \left|\frac{\Delta^k u_i}{h^k}\right|_1^p\right)^{1/p}.
$$

**Lemma 2.3 (Zhou (1984)).** Let $u_h = \{u_j = 0, \pm 1, \pm 2, \ldots, \pm J, \ldots\}$, $v_h = \{v_j = 0, \pm 1, \pm 2, \ldots, \pm J, \ldots\}$ such that $u_{j+1} = u_j$, $v_{j+1} = v_j$. We have

(i) $\sum_{j=0}^{J-1} u_j \Delta_+ v_j = -\sum_{j=1}^{J} v_j \Delta_- u_j$,

(ii) $\sum_{j=1}^{J} u_j \Delta_+ \Delta_- v_j = -\sum_{j=0}^{J-1} \Delta_+ u_j \Delta_+ v_j$,

(iii) $\Delta_+(u_j v_j) = u_{j+1} \Delta_+ v_j + v_{j+1} \Delta_+ u_j$,

where $\Delta_+, \Delta_-$ denote the forward and backward difference, respectively.

We use the differential–difference method to prove the local existence of smooth solutions of (1.7) and (1.8). We establish the following difference–differential equations:

$$
\frac{dZ_j}{dt} = \frac{\Delta_+ \Delta_- Z_j}{h^2} + \left|\frac{\Delta_+ Z_j}{h}\right|^2 Z_j + \frac{f_j Z_j \times \Delta_+ \Delta_- Z_j}{h^2} + \frac{\Delta_+ f_j Z_j \times \Delta_+ Z_j}{h}, \quad (2.1)
$$

$$
Z_j|_{t=0} = Z_{0j} = Z_0(jh), \quad (2.2)
$$

$$
Z_{j+1} = Z_j, \quad (2.3)
$$

where $j = 0, \pm 1, \ldots, \pm J, \ldots$, $h = 2D/J$, $J > 0$. 

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It is clear that the initial-value problem for ordinary differential equations (2.1)–(2.3) admits a local smooth solution. For such solution, we shall give some estimates uniformly in $h$ and then get a local smooth solution to problems (1.7) and (1.8). In this section, we always denote a smooth solution of (2.1)–(2.3) by $Z_j$.

**Lemma 2.4.** If $Z_0(x) \in H^1(Q)$, $(\partial f/\partial x) \in L^\infty(Q)$ then there are constants $T_0 > 0$, $C > 0$ independent of $h$ such that

$$
\sup_{0 \leq t \leq T_0} \|Z_h(t)\|_2 \leq C,
$$

(2.4)

$$
\sup_{0 \leq t \leq T_0} \|\delta Z_h(t)\|_2 \leq C.
$$

(2.5)

**Proof.** Multiplying (2.1) by $Z_j h$ and summing from $j=1$ to $J$, we have

$$
\frac{1}{2} \frac{d}{dt} \sum_{j=1}^J |Z_j|^2 h = - \sum_{j=0}^{J-1} \left| \frac{\Delta_+ Z_j}{h} \right|^2 h + \sum_{j=1}^J \left| \frac{\Delta_+ Z_j}{h^2} \right|^2 |Z_j|^2 h.
$$

It follows from lemma 2.2 that $\|Z_h\|_\infty \leq C(\|Z_h\|_2^{1/2} + (1/2D)\|Z_h\|_2)^{1/2}$.

Therefore, we have

$$
\frac{d}{dt} \|Z_h\|_2^2 + \|\delta Z_h\|_2^2 \leq C(\|Z_h\|_2^{1/2} + \|\delta Z_h\|_2^{1/2}).
$$

(2.6)

Moreover, multiplying (2.1) by $(\Delta_+ \Delta_- Z_j)/h$ and summing from $j=1$ to $J$, we get

$$
\sum_{j=1}^J \frac{\Delta_+ \Delta_- Z_j}{h} \cdot Z_j = \sum_{j=1}^J \left| \frac{\Delta_+ \Delta_- Z_j}{h^2} \right|^2 h + \sum_{j=1}^J \frac{\Delta_+ \Delta_- Z_j}{h} \cdot \left( \left| \frac{\Delta_+ Z_j}{h} \right|^2 Z_j \right) + \sum_{j=1}^J \frac{\Delta_+ f_j}{h} \left( Z_j \times \frac{\Delta_+ Z_j}{h} \right) \cdot \frac{\Delta_+ \Delta_- Z_j}{h^2} h.
$$

Therefore, one gets

$$
\frac{1}{2} \frac{d}{dt} \sum_{j=0}^{J-1} \left| \frac{\Delta_+ Z_j}{h} \right|^2 h + \sum_{j=1}^J \left| \frac{\Delta_+ \Delta_- Z_j}{h^2} \right|^2 h \leq \frac{1}{4} \sum_{j=1}^J \left| \frac{\Delta_+ \Delta_- Z_j}{h^2} \right|^2 h + C \max_{1 \leq j \leq J} |Z_j|^2 \sum_{j=1}^J \left| \frac{\Delta_+ Z_j}{h} \right|^4 h + C \max_{1 \leq j \leq J} |Z_j|^2 \sum_{j=1}^J \left| \frac{\Delta_+ Z_j}{h} \right|^2 h.
$$

Applying lemma 2.2, we have

$$
\|Z_h\|_\infty \leq C(\|Z_h\|_2^{3/4}(\|\delta^2 Z_h\|_2 + \|Z_h\|))^{1/4},
$$

$$
\|\delta Z_h\|_4 \leq C(\|\delta Z_h\|_2^{3/4}(\|\delta^2 Z_h\|_2 + \|\delta Z_h\|_2))^{1/4}.
$$
Then, it follows from Young’s inequality that
\[
\frac{d}{dt} \left( \|\delta Z_h\|_2^2 + \|\delta^2 Z_h\|_2^2 \right) \leq C \left( 1 + \|Z_h\|_2^8 + \|\delta Z_h\|_2^8 \right). \tag{2.7}
\]
Hence, putting (2.6) and (2.7) together, we have
\[
\frac{d}{dt} \left( \|\delta Z_h\|_2^2 + \|\delta Z_h\|_2^2 \right) \leq C \left( 1 + \|Z_h\|_2^2 + \|\delta Z_h\|_2^2 \right). \tag{2.8}
\]
This inequality, combined with Gronwall’s inequality, implies that there are constants \(T_0, C > 0\), independent of \(h\) such that
\[
\|Z_h(t)\|_2 + \|\delta Z_h(t)\|_2 \leq C, \quad \forall t \in [0, T_0], \quad \int_0^{T_0} \|\delta^2 Z_h(t)\|_2^2 dt \leq C.
\]
Lemma 2.4 is proved.

**Corollary 2.5.** Under the conditions in lemma 2.4 and \(f(x, t) \in L^\infty(Q)\), we have, for some constant \(C\) independent of \(h\),
\[
\sup_{0 \leq t \leq T_0; 1 \leq j \leq J} \|Z_j\| \leq C. \tag{2.9}
\]

**Lemma 2.6.** If \(Z_0(x) \in H^2(Q), f(x, t), (\partial f/\partial x) \in L^\infty(Q), (\partial^2 f/\partial x^2) \in L^\infty(Q)\), then there are constants \(T_0 > 0, C > 0\) independent of \(h\) such that
\[
\sup_{0 \leq t \leq T_0} \|\delta^2 Z_h(t)\|_2 \leq C, \tag{2.10}
\]
\[
\int_0^{T_0} \|\delta^3 Z_h(t)\|_2^2 dt \leq C.
\]

**Proof.** It follows from (2.1) that
\[
\frac{d}{dt} \Delta^+ Z_j = \frac{\Delta^+ \Delta^- Z_j}{h^2} + \Delta^+ \left( \frac{\Delta^+ Z_j}{h} \right) Z_j + f_j \Delta^+ \left( \frac{Z_j \times \Delta^+ \Delta^- Z_j}{h^2} \right) + \Delta^+ f_j \left( \frac{\Delta^+ \Delta^- Z_j}{h^2} \right) + \frac{\Delta^2 f_j}{h} \left( \frac{Z_j \times \Delta^+ \Delta^- Z_j}{h} \right).
\]
Multiplying this equality by \((\Delta^2 \Delta^- Z_j)/h^3\), summing it from \(j=1\) to \(J\) and noting that
\[
\sum_{j=1}^J \frac{\Delta^2 \Delta^- Z_j}{h^3} \cdot \Delta^+ Z_j = -\frac{1}{2} \frac{d}{dt} \sum_{j=0}^{J-1} \left| \frac{\Delta^2 Z_j}{h^2} \right|^2 h,
\]
\[
\sum_{j=1}^J \frac{\Delta^2 \Delta^- Z_j}{h^3} \cdot \Delta^+ f_j \left( \frac{Z_j \times \Delta^+ \Delta^- Z_j}{h^2} \right) \leq \frac{1}{10} \sum_{j=1}^J \left| \frac{\Delta^2 \Delta^- Z_j}{h^3} \right|^2 h
\]
\[
+ C \max_{1 \leq j \leq J} \|Z_j\| \sum_{j=1}^J \left| \frac{\Delta^2 \Delta^- Z_j}{h^2} \right|^2 h,
\]

the others can be given in a similar way. By the following inequalities
\[ \| \delta Z_h \|_\infty \leq C \| \delta Z_h \|_2^{1/2} (\| \delta^2 Z_h \|_2 + \| \delta Z_h \|_2)^{1/2}, \]
\[ \| \delta Z_h \|_6 \leq C \| \delta Z_h \|_2^{2/3} (\| \delta^2 Z_h \|_2 + \| \delta Z_h \|_2)^{1/3}, \]
we can get
\[ \frac{d}{dt} \sum_{j=0}^{J-1} \left| \frac{\Delta_+ Z_j}{h^2} \right|^2 h + \sum_{j=0}^{J-1} \left| \frac{\Delta_+ \Delta_- Z_j}{h^3} \right|^2 h \leq C \left( \sum_{j=0}^{J-1} \left| \frac{\Delta_+ Z_j}{h^2} \right|^2 h \right)^3. \]

Lemma 2.6 follows from Gronwall’s inequality. \[ \Box \]

**Corollary 2.7.** Under the conditions in lemma 2.6, we have, for some \( C \) independent of \( h \),
\[ \sup_{0 \leq t \leq T_0, 1 \leq j \leq J} \left| \frac{\Delta_+ Z_j(t)}{h} \right| \leq C, \quad (2.11) \]
\[ \int_0^{T_0} \| \delta Z_{ht}(t) \|_2^2 dt \leq C. \quad (2.12) \]

We will give the following lemma.

**Lemma 2.8.** If \( Z_0(x) \in H^3(Q), f(x, t), (\partial_t f/\partial x^i) \in L^\infty(Q), j = 1, 2, 3 \), then there are constants \( T_0 > 0, C > 0 \) independent of \( h \) such that
\[ \sup_{0 \leq t \leq T_0} \| \delta^3 Z_h(t) \|_2 \leq C, \quad (2.13) \]
\[ \int_0^{T_0} \| \delta^4 Z_h(t) \|_2 \leq C. \quad (2.14) \]

**Proof.** From (2.1) we have
\[ \sum_{j=1}^J \frac{\Delta_+^3 \Delta_- Z_j}{h^4} \cdot \frac{\Delta_+^2 Z_{jl}}{h^2} h \]
\[ = \sum_{j=1}^J \left| \frac{\Delta_+^3 \Delta_- Z_j}{h^4} \right|^2 h + \sum_{j=1}^J \left| \frac{\Delta_+ Z_{j+1}^2}{h^2} \right|^2 \frac{\Delta_+ Z_j}{h^2} \cdot \frac{\Delta_+^3 \Delta_- Z_j}{h^4} h \]
\[ + \sum_{j=1}^J \left( \frac{\Delta_+ Z_{j+2}}{h^4} \cdot \frac{\Delta_+ Z_{j+2}}{h^2} \right) \frac{\Delta_+ Z_j}{h^2} \cdot \frac{\Delta_+^3 \Delta_- Z_j}{h^4} h \]
\[ + \sum_{j=1}^J \left( \frac{\Delta_+ Z_{j+2}}{h^4} \cdot \frac{\Delta_+^2 Z_j}{h^2} \right) \frac{\Delta_+ Z_{j+1}}{h^4} \cdot \frac{\Delta_+ Z_{j+1}}{h^4} \frac{\Delta_+ Z_j}{h^4} \frac{\Delta_+^3 \Delta_- Z_j}{h^4} h \]
\[ + \sum_{j=1}^J \left( \frac{\Delta_+ Z_{j+2}}{h^4} \cdot \frac{\Delta_+ Z_{j+1}}{h^4} \right) \frac{\Delta_+ Z_j}{h^4} \cdot \frac{\Delta_+^3 \Delta_- Z_j}{h^4} h \]
\[ + \sum_{j=1}^J \left( \frac{\Delta_+ Z_{j+1}}{h^4} \cdot \frac{\Delta_+^2 Z_j}{h^2} \right) \frac{\Delta_+ Z_{j+1}}{h^4} \cdot \frac{\Delta_+^3 \Delta_- Z_j}{h^4} h \]

\[ + \sum_{j=1}^J \left( \frac{\Delta_+ Z_{j+1}}{h^4} \cdot \frac{\Delta_+^2 Z_{j+1}}{h^2} \right) \left( \frac{\Delta_+ Z_j}{h^4} \cdot \frac{\Delta_+^3 \Delta_- Z_j}{h^4} \right). \]
From lemma 2.2, we have
\[ \frac{\Delta^2 Z_{j+1}}{h} = \frac{\Delta^2 Z_j}{h^2} + \frac{\Delta_+ Z_j}{h^3} \left( Z_j \cdot \frac{\Delta^3 Z_{j+1}}{h^4} \right) \]
\[ + 2 \sum_{j=1}^{J} \frac{\Delta + Z_{j+1}}{h} \left( \Delta^2 Z_j \times \frac{\Delta^2 Z_j}{h^3} \right) \cdot \frac{\Delta^2 Z_j}{h^4} h \]
\[ + \sum_{j=1}^{J} \frac{\Delta + j + 1}{h^2} \left( \Delta + j \times \frac{\Delta^2 Z_j}{h^3} \right) \cdot \frac{\Delta^2 Z_j}{h^4} h \]
\[ + \sum_{j=1}^{J} \frac{\Delta + j + 2}{h^2} \left( Z_{j+2} \times \frac{\Delta^3 Z_j}{h^3} \right) \cdot \frac{\Delta^2 Z_j}{h^4} h \]
\[ + \sum_{j=1}^{J} \frac{\Delta + j + 3}{h^2} \left( Z_{j+3} \times \frac{\Delta^3 Z_j}{h^3} \right) \cdot \frac{\Delta^2 Z_j}{h^4} h \]
\[ + \sum_{j=1}^{J} \frac{\Delta^3 Z_{j+1}}{h^2} \left( Z_{j+1} \times \frac{\Delta^3 Z_{j+1}}{h^3} \right) \cdot \frac{\Delta^3 Z_{j+1}}{h^4} h \]
\[ + \sum_{j=1}^{J} \frac{\Delta^3 Z_j}{h^3} \left( Z_j \times \frac{\Delta^3 Z_j}{h^3} \right) \cdot \frac{\Delta^3 Z_j}{h^4} h. \]

From lemma 2.2, we have \( \| \delta^2 Z_h \| \leq C \| Z_h \|^{1/4} \left( \| \delta^3 Z_h \| + \| Z_h \| \right)^{3/4} \).

By lemmas 2.3, 2.4 and 2.6 and a similar method to lemma 2.6, we can have
\[ \frac{1}{2} \frac{d}{dt} \sum_{j=0}^{J-1} \left| \frac{\Delta^3 Z_j}{h^3} \right|^2 h + \frac{1}{2} \sum_{j=1}^{J} \left| \frac{\Delta^3 Z_j}{h^4} \right|^2 h \leq C \sum_{j=0}^{J-1} \left| \frac{\Delta^3 Z_j}{h^3} \right|^2 h + C. \]

Lemma 2.8 follows from Gronwall’s inequality. \( \square \)
By the induction method we have the following lemma.

**Lemma 2.9.** If \( Z_0(x) \in H^k(Q) \), \( f(x, t) \in L^\infty(Q) \), \((\partial^j f/\partial x^j) \in L^\infty(Q), j = 1, 2, \ldots, k \), then there are constants \( T_0 > 0, C > 0 \) independent of \( h \) such that

\[
\sup_{0 \leq t \leq T_0} \| \delta^k Z_{ht} \|_2 \leq C, \quad (k \geq 2),
\]

\[
\sup_{0 \leq t \leq T_0} \| \delta^{k-4} Z_{ht} \|_2 \leq C, \quad (k \geq 4).
\]

From lemma 2.9, we obtain the \textit{a priori} estimates for solutions to the differential–difference equations (2.1)–(2.3). Using the same method as in Zhou \textit{et al.} (1991), we conclude that there exists a constant \( T_0 > 0 \) such that problems (1.7) and (1.8) admit a smooth solution in \( Q \times [0, T_0] \). This result is stated as follows.

**Theorem 2.10.** Let \( \varepsilon > 0 \), \( Z_0(x) \in H^k(Q) \), \( f(x, t) \in L^\infty(Q) \), \((\partial^j f/\partial x^j) \in L^\infty(Q), j = 1, 2, \ldots, k \), \( f(x, t) \geq f_0 > 0 \). Then (1.7) and (1.8) admit a local smooth solution \( Z(x, t) \) in \([0, T_0]\) with \( T_0 \) depending on \( k \):

\[
Z(x, t) \in \left( \bigcap_{s=0}^{[k/2]} W^8_{\infty}(0, T_0; H^{k-2s}(Q)) \right) \cap \left( \bigcap_{s=0}^{[(k+1)/2]} H^s(0, T_0; H^{k+1-2s}(Q)) \right).
\]

### 3. \( \varepsilon > 0 \): global solution and uniform estimates

In §2, we have obtained a local smooth solutions for (1.7) and (1.8) when \( \varepsilon > 0 \) is fixed. In this section, we intend to prove the global existence of smooth solutions to problems (1.7) and (1.8) for fixed \( \varepsilon > 0 \) by deriving the global (in time) estimates. To meet the need to send \( \varepsilon \) to zero, we want these estimates to be both global in time and uniform in \( \varepsilon \). The difficulty in the uniform estimate is overcome by constructing some new energy laws and by using the fact that \( Z, Z_x, Z \times Z_x \) is a base of \( R^3 \), seeing \( L^\infty(0, T; H^2) \) and \( L^\infty(0, T; H^3) \) estimates below.

In the following, we always suppose \( Z(x, t) \) is a global smooth solution of problems (1.7) and (1.8) for \( \varepsilon > 0 \). We intend to derive the following global and uniform estimates.

**Lemma 3.1.** If \( Z_0(x) \in H^1 \) and \( Z(x, t) \) is a global smooth solution of problems (1.7) and (1.8), then we have

\[
|Z(x, t)| = 1, \quad \forall (x, t) \in R^1 \times [0, +\infty).
\]

\[\text{Proof.} \] Multiplying (1.9) by \( Z(x, t) \), we have \( Z(x, t) \cdot Z_t(x, t) = 0 \). This implies the conclusion of lemma 3.1.

**Lemma 3.2.** Under the same conditions of lemma 3.1 and \( \partial f/\partial t, (\partial f/\partial x) \in L^\infty(Q) \), we have, for any given \( T > 0 \), there exists \( C > 0 \) independent of \( \varepsilon \) and \( D \),

\[
\sup_{0 \leq t \leq T} \| Z_x(\cdot, t) \|_2 \leq C.
\]
Proof. From (1.9), we have

$$Z_{xt} = -\varepsilon (Z \times (Z \times Z_{xx}))_x + \frac{\partial^2 f}{\partial x^2} Z \times Z_x + 2 \frac{\partial f}{\partial x} Z \times Z_{xx} + fZ_x \times Z_{xx} + fZ \times Z_{xxx}. $$

By \((A \times (B \times C)) \cdot B = (A \times B) \cdot (C \times B), we get

$$\int_\Omega fZ_x \cdot Z_{xt} dx = \varepsilon \int_\Omega \frac{\partial f}{\partial x} (Z \times (Z \times Z_{xx})) \cdot Z_x dx - \varepsilon \int_\Omega f|Z \times Z_{xx}|^2 dx.$$ 

Therefore, one gets

$$\frac{1}{2} \frac{d}{dt} \int_\Omega f|Z_x|^2 dx + \varepsilon \int_\Omega f|Z \times Z_{xx}|^2 dx = \frac{1}{2} \int_\Omega \frac{\partial f}{\partial t} |Z_x|^2 dx + \varepsilon \int_\Omega \frac{\partial f}{\partial x} (Z \times (Z \times Z_{xx})) \cdot Z_x dx$$

$$\leq C \int_\Omega f|Z_x|^2 dx + \frac{\varepsilon}{2} \int_\Omega f|Z \times Z_{xx}|^2 dx.$$ 

It is easy to see that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega f|Z_x|^2 dx \leq C \int_\Omega f|Z_x|^2 dx.$$ 

Then lemma 3.2 follows. \hfill \Box

Lemma 3.3. Let \(Z_0(x) \in H^2(\Omega), f(x, t), \frac{\partial f}{\partial t}, (\frac{\partial^j f}{\partial x^j} \in L^\infty(\Omega), j = 1, 2, 3, \) then for any given \(T > 0,\) there are constants \(C > 0\) independent of \(\varepsilon, D,\) such that

$$\sup_{0 \leq t \leq T} \|Z_{xx}(\cdot, t)\|_2 \leq C; \quad (3.3)$$

$$\sup_{0 \leq t \leq T} \|Z_t(\cdot, t)\|_2 \leq C. \quad (3.4)$$

Proof. In the proof, we shall use the following identities which follow from the fact \(|Z(x, t)| = 1:\)

$$Z \cdot Z_t = 0, \quad Z \cdot Z_x = 0, \quad Z \cdot Z_{xx} = -|Z_x|^2, \quad Z \cdot Z_{xxx} = -3Z_x \cdot Z_{xx}. \quad (3.5)$$

It follows from (1.7) that

$$Z_{xt} = \varepsilon Z_{xxx} + \varepsilon |Z_x|^2 Z_{xx} + 2\varepsilon (Z_x \cdot Z_{xx}) Z_x + 2\varepsilon [(Z_x \cdot Z_{xx}) Z_x$$

$$+ \frac{\partial^3 f}{\partial x^3} Z \times Z_x + 3 \frac{\partial^2 f}{\partial x^2} Z \times Z_{xx} + 3 \frac{\partial f}{\partial x} Z_x \times Z_{xx} + 3 \frac{\partial f}{\partial x} Z \times Z_{xxx}$$

$$+ 2f Z_x \times Z_{xx} + f Z \times Z_{xxx}. \quad (3.6)$$

From (3.6) we get

$$\frac{1}{2} \frac{d}{dt} \int_\Omega f^2 |Z_{xx}|^2 dx = \int_\Omega f \frac{\partial f}{\partial t} |Z_{xx}|^2 dx - \varepsilon \int_\Omega f^2 |Z_{xxx}|^2 dx + \varepsilon \int_\Omega f^2 |Z_x|^2 |Z_{xx}|^2 dx$$

$$+ 8\varepsilon \int_\Omega f^2 |Z_x \cdot Z_{xx}|^2 dx - 4\varepsilon \int_\Omega f \frac{\partial f}{\partial x} (Z_x \cdot Z_{xx}) (Z \cdot Z_{xx}) dx$$

$$+ \int_\Omega f^2 \frac{\partial^3 f}{\partial x^3} (Z \times Z_x) \cdot Z_{xx} dx - 2\varepsilon \int_\Omega f \frac{\partial f}{\partial x} Z_{xx} \cdot Z_{xxx} dx$$

$$- \int_\Omega f^3 (Z_x \times Z_{xx}) \cdot Z_{xxx} dx. \quad (3.7)$$
Note that, if $|Z_x| \neq 0$, then the vectors $Z, Z_x, Z \times Z_x$ form an orthogonal basis of $R^3$. Let

$$Z_{xx} = \alpha Z + \beta Z_x + \gamma Z \times Z_x,$$

and it is easy to see that

$$\alpha = -|Z_x|^2, \quad \beta = \frac{Z_x \cdot Z_{xx}}{|Z_x|^2}, \quad \gamma = \frac{(Z \times Z_x) \cdot Z_{xx}}{|Z_x|^2}.$$ 

Therefore, we have

$$- \int_{\Omega} f^3 (Z_x \times Z_{xx}) \cdot Z_{xxx} \, dx$$

$$= - \int_{\Omega} f^3 |Z_x|^2 (Z \times Z_x) \cdot Z_{xxx} \, dx - \int_{\Omega} f^3 [(Z \times Z_x) \cdot Z_{xx}] \left(-\frac{3}{2} |Z_x|^2 \right) \, dx$$

$$= -\frac{5}{2} \int_{\Omega} f^3 |Z_x|^2 (Z \times Z_x) \cdot Z_{xxx} \, dx - \frac{9}{2} \int_{\Omega} f^3 \frac{\partial f}{\partial x} |Z_x|^2 (Z \times Z_x) \cdot Z_{xx} \, dx,$$

where we have used the fact that $A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$.

Combining (3.7) with (3.8), we get

$$2 \frac{d}{dt} \int_{\Omega} f^2 |Z_{xx}|^2 \, dx + 4\varepsilon \int_{\Omega} f^2 |Z_{xxx}|^2 \, dx$$

$$= -8\varepsilon \int_{\Omega} f \frac{\partial f}{\partial x} Z_{xx} \cdot Z_{xxx} \, dx + 4\varepsilon \int_{\Omega} f^2 |Z_x|^2 |Z_{xx}|^2 \, dx + 32\varepsilon \int_{\Omega} f^2 |Z_x \cdot Z_{xx}|^2 \, dx$$

$$- 16\varepsilon \int_{\Omega} f \frac{\partial f}{\partial x} (Z_x \cdot Z_{xx})(Z_x \cdot Z_{xxx}) \, dx + 4 \int_{\Omega} f \frac{\partial f}{\partial t} |Z_{xx}|^2 \, dx$$

$$+ 4 \int_{\Omega} f^2 \frac{\partial^3 f}{\partial x^3} (Z \times Z_x) \cdot Z_{xx} \, dx - 10 \int_{\Omega} f^3 |Z_x|^2 (Z \times Z_x) \cdot Z_{xxx} \, dx$$

$$- 18 \int_{\Omega} f^2 \frac{\partial f}{\partial x} |Z_x|^2 (Z \times Z_x) \cdot Z_{xx} \, dx.$$

On the other hand, from (1.7) we get

$$\frac{5}{2} \frac{d}{dt} \int_{\Omega} f^2 |Z_x|^4 \, dx = 5 \int_{\Omega} f \frac{\partial f}{\partial t} |Z_x|^4 \, dx + 10\varepsilon \int_{\Omega} f^2 |Z_x|^2 Z_x \cdot Z_{xxx} \, dx$$

$$+ 10\varepsilon \int_{\Omega} f^2 |Z_x|^6 \, dx - 20 \int_{\Omega} f^2 \frac{\partial f}{\partial x} |Z_x|^2 (Z \times Z_x) \cdot Z_{xx} \, dx$$

$$- 10 \int_{\Omega} f^3 |Z_x|^2 (Z \times Z_x) \cdot Z_{xxx} \, dx.$$
Putting (3.9) and (3.10) together and using (1.5), we have

\[
\begin{align*}
2 \frac{d}{dt} \int_\Omega f^2 |Z_{xx}|^2 dx + 4 \varepsilon \int_\Omega f^2 |Z_{xxx}|^2 dx + 10 \varepsilon \int_\Omega f |Z_x|^6 dx - \frac{5}{2} \frac{d}{dt} \int_\Omega f^2 |Z_x|^4 dx & \\
= -5 \int_\Omega f \frac{\partial f}{\partial t} |Z_x|^4 dx + 4 \int_\Omega f \frac{\partial f}{\partial t} |Z_{xx}|^2 dx - 8 \varepsilon \int_\Omega f \frac{\partial f}{\partial x} Z_{xx} \cdot Z_{xxx} dx & \\
+ 4 \varepsilon \int_\Omega f^2 |Z_x|^2 |Z_{xx}|^2 dx + 32 \varepsilon \int_\Omega f^2 |Z_x \cdot Z_{xx}|^2 dx + 16 \varepsilon \int_\Omega f \frac{\partial f}{\partial x} |Z_x|^2 Z_x \cdot Z_{xx} dx & \\
- 10 \varepsilon \int_\Omega f^2 |Z_x|^2 Z_x \cdot Z_{xxx} dx + 2 \int_\Omega f^3 \frac{\partial f}{\partial x} |Z_x|^2 (Z \times Z_x) \cdot Z_{xx} dx & \\
+ 4 \int_\Omega f^2 \frac{\partial^3 f}{\partial x^3} (Z \times Z_x) \cdot Z_{xx} dx.
\end{align*}
\]

\[ (3.11) \]

By lemmas 2.1 and 3.1, we have

\[
4 \varepsilon \int_\Omega f^2 |Z_x|^2 |Z_{xx}|^2 dx + 32 \varepsilon \int_\Omega f^2 |Z_x \cdot Z_{xx}|^2 dx \\
\leq C \delta_1 \varepsilon \int_\Omega f^2 |Z_x|^6 dx + C \varepsilon \delta_2 \int_\Omega f^2 |Z_{xxx}|^2 dx + C.
\]

Letting \( \delta_1 = 2/C, \delta_2 = 1/C \), we get

\[
\begin{align*}
2 \frac{d}{dt} \int_\Omega f^2 |Z_{xx}|^2 dx + 3 \varepsilon \int_\Omega f^2 |Z_{xxx}|^2 dx + 8 \varepsilon \int_\Omega f^2 |Z_x|^6 dx - \frac{5}{2} \frac{d}{dt} \int_\Omega f^2 |Z_x|^4 dx & \\
\leq 4 \int_\Omega f \frac{\partial f}{\partial t} |Z_{xx}|^2 dx - 5 \int_\Omega f \frac{\partial f}{\partial t} |Z_x|^4 dx - 8 \varepsilon \int_\Omega f \frac{\partial f}{\partial x} Z_{xx} \cdot Z_{xxx} dx & \\
- 10 \varepsilon \int_\Omega f^2 |Z_x|^2 Z_x \cdot Z_{xxx} dx + 16 \varepsilon \int_\Omega f \frac{\partial f}{\partial x} |Z_x|^2 Z_x \cdot Z_{xx} dx & \\
+ 2 \int_\Omega f^3 \frac{\partial f}{\partial x} |Z_x|^2 (Z \times Z_x) \cdot Z_{xx} dx + 4 \int_\Omega f^2 \frac{\partial^3 f}{\partial x^3} (Z \times Z_x) \cdot Z_{xx} dx + C & \\
\leq C \int_\Omega f^2 |Z_{xx}|^2 dx + 6 \varepsilon \int_\Omega f^2 |Z_x|^6 dx + \varepsilon \int_\Omega f^2 |Z_{xxx}|^2 dx + C,
\end{align*}
\]

\[ (3.12) \]

where we have used the fact that \( \|Z_x\|_6 \leq \|Z_x\|_2^{2/3} \|Z_{xx}\|_2^{1/3} \).

Therefore, one gets

\[
\frac{d}{dt} \int_\Omega f^2 |Z_{xx}|^2 dx \leq C \frac{d}{dt} \int_\Omega f^2 |Z_x|^4 dx + C \int_\Omega f^2 |Z_x|^2 dx + C.
\]

\[ (3.13) \]
Integrating (3.13), by lemma 3.2 and \( \|Z_x\|_4 \leq C \|Z_x\|_2^{3/4} \|Z_{xx}\|_2^{1/4} \), we have
\[
\int_0^T f^2 |Z_{xx}|^2 dx \leq C \int_0^T f^2 |Z_{xx}|^2 dx dt + C.
\]

The conclusion of lemma 3.3 follows from this inequality and Gronwall’s inequality.

Similarly, we have the following lemma:

**Lemma 3.4.** Let \( Z_0(x) \in H^3(\Omega) \), \( f(x, t), \partial f/\partial t, (\partial^i f/\partial x^i) \in L^\infty(\Omega), j = 1, 2, 3, 4 \). For any given \( T > 0 \), there are \( C > 0 \) independent of \( \varepsilon, D \) such that
\[
\sup_{0 \leq t \leq T} \|Z_{xxx}(\cdot, t)\|_2 \leq C,
\]
\[
\sup_{0 \leq t \leq T} \|Z_{xt}(\cdot, t)\|_2 \leq C.
\]

**Proof.** It follows from (3.6) that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega f^3 |Z_{xxx}|^2 dx
\]
\[
= \frac{3}{2} \int_\Omega f \frac{\partial f}{\partial t} |Z_{xxx}|^2 dx - \varepsilon \int_\Omega f^3 |Z_{xxx}|^2 dx - 3 \varepsilon \int_\Omega f^2 \frac{\partial f}{\partial x} Z_{xx} \cdot Z_{xxx} dx
\]
\[
+ \varepsilon \int_\Omega f^3 |Z_x|^2 |Z_{xxx}|^2 dx + 4 \varepsilon \int_\Omega f^3 (Z_x \cdot Z_{xx}) (Z_{xx} \cdot Z_{xxx}) dx
\]
\[
+ 2 \varepsilon \int_\Omega f^3 |Z_{xx}|^2 (Z_x \cdot Z_{xxx}) dx + 2 \varepsilon \int_\Omega f^3 |Z_x \cdot Z_{xxx}|^2 dx
\]
\[
- 2 \varepsilon \int_\Omega f^3 |Z_{xx}|^2 Z + (Z_x \cdot Z_{xxx}) Z + (Z_x \cdot Z_{xx}) Z_x \cdot Z_{xxx} dx
\]
\[
- 6 \varepsilon \int_\Omega f^2 \frac{\partial f}{\partial x} |Z_{xx}|^2 Z + (Z_x \cdot Z_{xxx}) Z + (Z_x \cdot Z_{xx}) Z_x \cdot Z_{xxx} dx
\]
\[
+ \int_\Omega f^3 \frac{\partial^3 f}{\partial x^3} (Z \times Z_x) \cdot Z_{xxx} dx + 4 \int_\Omega f^3 \frac{\partial^3 f}{\partial x^2} (Z \times Z_{xx}) \cdot Z_{xxx} dx
\]
\[
+ 6 \int_\Omega f^3 \frac{\partial^2 f}{\partial x^2} (Z_x \times Z_{xx}) \cdot Z_{xxx} dx - 2 \int_\Omega f^4 (Z_x \times Z_{xxx}) \cdot Z_{xxxx} dx. \tag{3.16}
\]

Then, if \( |Z_x| \neq 0 \), let \( Z_{xxx} = \alpha' Z + \beta' Z_x + \gamma' Z \times Z_x \).

It is easy to see that
\[
\alpha' = -3 Z_x \cdot Z_{xx}, \quad \beta' = \frac{Z_x \cdot Z_{xxx}}{|Z_x|^2}, \quad \gamma' = \frac{(Z \times Z_x) \cdot Z_{xxx}}{|Z_x|^2}.
\]
Then
\[
-2\int_{\Omega} f^4(Z_x \times Z_{xxxx}) \cdot Z_{xxxx} \, dx
\]
\[
= 24 \int_{\Omega} f^3 \frac{\partial f}{\partial x} (Z_x \cdot Z_{xx})(Z \times Z_x) \cdot Z_{xxxx} \, dx
\]
\[
+ 12 \int_{\Omega} f^4 |Z_{xx}|^2 (Z \times Z_x) \cdot Z_{xxxx} \, dx
\]
\[
+ 6 \int_{\Omega} f^4 (Z_x \cdot Z_{xx})(Z \times Z_x) \cdot Z_{xxxx} \, dx
\]
\[
+ 6 \int_{\Omega} f^4 (Z_x \cdot Z_{xx})(Z \times Z_x) \cdot Z_{xxxx} \, dx
\]
\[
+ 8 \int_{\Omega} f^4 ((Z \times Z_x) \cdot Z_{xx})(Z \times Z_{xx}) \, dx \leq C \int_{\Omega} f^3 |Z_{xx}|^2 \, dx + C,
\]
(3.17)

where we have used the fact \( Z \cdot Z_{xxxx} = -3|Z_{xx}|^2 - 4Z_x \cdot Z_{xx} \).

Putting (3.16) and (3.17) together, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} f^3 |Z_{xx}|^2 \, dx + \varepsilon \int_{\Omega} f^3 |Z_{xxxx}|^2 \, dx \leq C \int_{\Omega} f^3 |Z_{xx}|^2 \, dx + \varepsilon \int_{\Omega} f^3 |Z_{xxxx}|^2 \, dx + C.
\]

Employing Gronwall's inequality, we obtain (3.14). Then, one can obtain (3.15) easily.

By induction, we have the following lemma.

**Lemma 3.5.** Let \( Z_0(x) \in H^k(\Omega) \), \( f(x, t), \frac{\partial f}{\partial t}, (\frac{\partial^j f}{\partial x^j}) \in L^\infty(Q), j = 1, 2, \ldots, k + 1 \). For any given \( T > 0 \), there is \( C > 0 \) independent of \( \varepsilon, D \) such that
\[
\sup_{0 \leq t \leq T} \| \partial_x^s \partial_{xx}^{k-2s} Z(\cdot, t) \|_2 \leq C, \ 0 \leq s \leq [k/2].
\]
(3.18)

Combining the local existence obtained in §2 and the global in time estimates in lemmas 3.1–3.5, we can get the global existence of smooth solutions to problems (1.7) and (1.8) for fixed \( \varepsilon > 0 \) in the following sense.

**Theorem 3.6.** Let \( Z_0(x) \in H^k(\Omega) \), \( f(x, t), \frac{\partial f}{\partial t}, (\frac{\partial^j f}{\partial x^j}) \in L^\infty(Q), j = 1, 2, \ldots, k + 1 \), \( f(x, t) \geq f_0 > 0 \). Then, for any given \( T > 0 \), problems (1.7) and (1.8) admit at least one smooth solution \( Z(x, t) \) in \([0, T] \):
\[
Z(x, t) \in \bigcap_{s=0}^{[k/2]} W^s(0, T; H^{k-2s}(\Omega)).
\]
(3.19)

We should note that the \textit{a priori} estimates in lemmas 3.1–3.5 are uniform in \( \varepsilon \).

4. \( \varepsilon = 0 \): global solution and uniqueness

In §3, we have obtained a global smooth solution for (1.7) and (1.8) for fixed \( \varepsilon > 0 \), and the estimates are all uniform in \( \varepsilon \). These uniform estimates enable us to pass to the limit \( \varepsilon \to 0 \) in equation (1.7), and then to get the global smooth solutions of problems (1.1) and (1.2). Therefore, we have the following theorem.
Theorem 4.1. Let $Z_0(x) \in H^k(\Omega), f(x,t), \partial f/\partial t, (\partial^j f/\partial x^j) \in L^\infty(\Omega), j = 1, 2, \ldots, k + 1, f(x,t) \geq f_0 > 0$. Then (1.1) and (1.2) admit a global smooth solution $Z(x,t)$:

$$Z(x,t) \in \bigcap_{s=0}^{[k/2]} W^s_x(0, \infty; H^{k-2s}(\Omega)).$$

On the other hand, all the estimates in §3 are independent of $D$. Thus, by sending $D$ to $\infty$, we get the global existence of smooth solution to the Cauchy problem of (1.1).

Theorem 4.2. Let $Z_0(x) \in H^k(\Omega), f(x,t), \partial f/\partial t, (\partial^j f/\partial x^j) \in L^\infty(\Omega), j = 1, 2, \ldots, k + 1, f(x,t) \geq f_0 > 0$. Then, the Cauchy problem of (1.1) admits a global smooth solution $Z(x,t)$:

$$Z(x,t) \in \bigcap_{s=0}^{[k/2]} W^s_x(0, \infty; H^{k-2s}(\Omega)).$$

Remark 4.3. For any given $\varepsilon > 0$, it is not difficult to see that the problems (1.7) and (1.8) (or the Cauchy problem of (1.7)) admit at least one global smooth solution.

Now, we turn to prove that the global smooth solution of (1.1) and (1.2) obtained in theorem 4.1 is unique. That is, we prove the following theorem.

Theorem 4.4. The global smooth solutions of (1.1) and (1.2) obtained in theorem 4.1 is unique.

Proof. First of all, we prove that the periodic problems (1.7) and (1.8) admit a unique global smooth solution. Let $Z_1(x,t)$ and $Z_2(x,t)$ be smooth solutions of (1.7) and (1.8). Let $W(x,t) = Z_1 - Z_2$.

Then, we have

$$W_t = \varepsilon W_{xx} + \varepsilon Z_2[(Z_{1x} + Z_{2x}) \cdot W] + \varepsilon |Z_{1x}|^2 W + f Z_2 \times W_{xx}$$

$$+ f W \times Z_{1xx} + \frac{\partial f}{\partial x} Z_2 \times W_x + \frac{\partial f}{\partial x} W \times Z_{1x}, \quad (4.1)$$

with homogeneous initial boundary conditions. Multiplying (4.1) by $W$, using Hölder’s inequality and noting that $Z_1, Z_2$ are smooth, we have

$$\frac{1}{2} \frac{d}{dt} \int_{-D}^{D} |W|^2 dx + \varepsilon \int_{-D}^{D} |W_x|^2 dx \leq \frac{\varepsilon}{4} \int_{-D}^{D} |W_{xx}|^2 dx + C \int_{-D}^{D} |W_x|^2 dx$$

$$+ C \int_{-D}^{D} |W|^2 dx. \quad (4.2)$$

On the other hand, multiplying (4.1) by $W_{xx}$, we have

$$\frac{1}{2} \frac{d}{dt} \int_{-D}^{D} |W_x|^2 dx + \varepsilon \int_{-D}^{D} |W_{xx}|^2 dx \leq \frac{\varepsilon}{4} \int_{-D}^{D} |W_{xx}|^2 dx + C \int_{-D}^{D} |W_x|^2 dx$$

$$+ C \int_{-D}^{D} |W|^2 dx. \quad (4.3)$$

Combining (4.2) and (4.3), using Gronwall’s inequality and noting that \( W(x,0) = 0, W_x(x,0) = 0 \), we can get the uniqueness when \( \varepsilon > 0 \).

In the second step, we let \( \varepsilon = 0 \) in (4.1) to get

\[
\frac{d}{dt} \int_{-D}^{D} |W|^2 \, dx - \int_{-D}^{D} \frac{\partial f}{\partial x} (Z_2 \times W) \cdot W_x \, dx + \int_{-D}^{D} f(Z_2 \times W) \cdot W_x \, dx \leq C \int_{-D}^{D} |W|^2 \, dx + C \int_{-D}^{D} |W_x|^2 \, dx.
\]

On the other hand, from (4.4) we have

\[
\frac{d}{dt} \int_{-D}^{D} |W|^2 \, dx = \frac{1}{2} \int_{-D}^{D} \frac{\partial f}{\partial x} (Z_2 \times W) \cdot W_x \, dx + \frac{1}{2} \int_{-D}^{D} f(Z_2 \times W) \cdot W_x \, dx + \int_{-D}^{D} f^2(Z_2 \times W) \cdot W_x \, dx
\]

\[
+ \int_{-D}^{D} f^2(Z_2 \times W) \cdot W_x \, dx + \int_{-D}^{D} f^2(Z_2 \times W) \cdot W_x \, dx + \int_{-D}^{D} f^2(Z_2 \times W) \cdot W_x \, dx \leq C \int_{-D}^{D} |W|^2 \, dx + C \int_{-D}^{D} |W_x|^2 \, dx.
\]

Putting (4.5) and (4.6) together, we get

\[
\frac{d}{dt} \left( \int_{-D}^{D} |W|^2 \, dx + \int_{-D}^{D} f|W_x|\, dx \right) \leq C \left( \int_{-D}^{D} |W|^2 \, dx + \int_{-D}^{D} f|W_x|^2 \, dx \right).
\]

Using Gronwall’s inequality and noting that \( W(x,0)=0, W_x(x,0)=0 \), we can obtain the conclusion to theorem 4.1.  

\[\blacksquare\]
5. Conclusion

In this paper, we obtain existence and uniqueness of global smooth solutions to the periodic initial problem of the one-dimensional inhomogeneous non-automorphic Landau–Lifshitz equation, using the differential–difference method and viscosity vanishing approach. First, we establish the existence and uniqueness of the smooth solution to (1.7) and (1.8) by differential–difference method. Secondly, we give some a priori estimates for the solution of the viscosity problem uniform in \( \varepsilon \) by constructing some new energy laws. Finally, we send \( \varepsilon \) to zero to obtain the existence of the smooth solution to problems (1.1) and (1.2). In this procedure, the usage of the orthogonal base \( \{ \mathbf{Z}, \mathbf{Z}_x, \mathbf{Z} \times \mathbf{Z}_x \} \) simplifies our proof.

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