Global solutions to one-dimensional compressible Navier–Stokes–Poisson equations with density-dependent viscosity

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(Received 12 November 2008; accepted 10 January 2009; published online 20 February 2009)

In this paper, we prove the global existence of weak solutions to one-dimensional compressible isentropic Navier–Stokes–Poisson equations with density-dependent viscosity and free boundaries. The initial density \( \rho_0 \in W^{1,2n} \) is bounded below by a positive constant, and the initial velocity \( u_0 \in L^{2n} \). In contrast to Jiang et al. [“Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity,” Methods Appl. Anal. 12, 239 (2005)], the Sobolev exponent \( n \) is less in this paper, and the viscosity coefficient \( \mu = \mu(\rho) \) is a general function of \( \rho \) including the cases \( \mu(\rho) = c_0 \rho^{\theta} \) (0 < \( \theta \) < 1) and \( \mu(\rho) = c_0 \), where \( c_0 \) is a positive constant. © 2009 American Institute of Physics. [DOI: 10.1063/1.3078384]

I. INTRODUCTION

In this paper, we consider the global existence of weak solutions to the following compressible isentropic Navier–Stokes–Poisson equations with density-dependent viscosity in one dimension,

\[
\rho_t + (\rho u)_x = 0,
\]

\[
(\rho u)_t + (\rho u^2 + P(\rho))_x + \rho \Phi_x = (\mu(\rho)u)_x,
\]

\[
\Phi_x = 4\pi g \left( \rho - \frac{1}{|M|} \int_{M} \rho dx \right), \quad (x,t) \in \Omega,
\]

with initial data

\[
(\rho,u)|_{t=0} = (\rho_0(x),u_0(x)), \quad \text{on}\ a \leq x \leq b,
\]

and the boundary conditions

\[
(\mu(\rho)u_x - P)|_{x=a(t),b(t)} = 0, \quad \Phi|_{x=a(t),b(t)} = 0 \quad \text{for} \ t \geq 0,
\]

where \( \Omega_t = \{ (x,t) | a(t) < x < b(t), t > 0 \} \), \( M_t = \{ x | a(t) < x < b(t) \} \). The unknown variables \( \rho = \rho(x,t) \), \( u = u(x,t) \), and \( \Phi = \Phi(x,t) \) denote, respectively, the density, velocity, and Newtonian gravitational potential. \( g > 0 \) is the gravitational constant. The pressure function \( P(\rho) = \rho^\gamma \) for \( \gamma > 1 \). The viscosity coefficient \( \mu = \mu(\rho) \in C([0,\infty)) \) satisfies

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\[\mu(\rho) \geq c_0, \quad \forall \rho \geq 1,\]
\[\mu(\rho) \geq c_0 \rho^\theta, \quad \forall 0 \leq \rho < 1 \quad \text{for} \ 0 < \theta < 1,\]  
where \(c_0\) is a positive constant. Moreover, \(a(t), b(t)\) are the free boundaries, satisfying

\[\frac{da(t)}{dt} = u(a(t), t), \quad \frac{db(t)}{dt} = u(b(t), t),\]

\[a(0) = a, \quad b(0) = b.\]

Physically, the Navier–Stokes–Poisson equations describe the motion of compressible viscous isentropic gas flow under the self-gravitational force.

When there is no Newtonian gravitational potential term, it is Navier–Stokes equations with density-dependent viscosity. For the case \(\mu(\rho) = c_0 \rho^\theta\), there are many results. Precisely, when the initial density was assumed to be connected to vacuum with discontinuities, the global existence of weak solutions for isentropic flow was proven by Okada et al. in Ref. 7 for \(0 < \theta < \frac{1}{2}\) and by Yang et al. in Ref. 13 for \(0 < \theta < \frac{1}{2}\) and by Jiang et al. in Ref. 5 for \(0 < \theta < 1\) with initial data \(\rho_0 \in W^{1,2n}, u_0 \in L^{2n}\), where \(n \in \mathbb{N}\) satisfies \((2n^2 - n) / (2n^2 + 2n - 1) > \theta\). Recently, Qin and Yao in Ref. 6 obtained the global existence of weak solutions for \(0 < \theta \leq 1\) with initial data \(\rho_0 \in W^{1,\infty}, u_0 \in L^\infty\).

The global existence of smooth solutions for nonisentropic flow may be referred to Refs. 4, 6, and 8. In Ref. 6, the viscosity coefficient \(\mu\) is assumed to be as a general function of \(\rho\) satisfying

\[0 < \mu_0 \leq \mu(\rho) \leq \mu_1 \quad \text{for} \ \rho \geq 0,\]

where \(\mu_0\) and \(\mu_1\) are two positive constants, which shows the viscosity is nondegenerate, even if the vacuum appears. In order to consider the case of degenerate viscosity as the appearance of vacuum, Jiang et al. in Ref. 4 and Qin and Yao discussed the case \(\mu(\rho) = \rho^\theta\) for \(\theta \in (0, \frac{1}{2}]\) and \(\theta \in (0, \frac{1}{2}]\), respectively. The viscosity coefficient \(\mu\) in Refs. 4–9 and 13 can be viewed as a special case of (1.4).

To our knowledge, there are few results about the Navier–Stokes–Poisson equations with density-dependent viscosity. The authors in Refs. 12 and 16 give some results about the case when the viscosity coefficient \(\mu\) is a constant and the boundaries are fixed. For more results about Navier–Stokes–Poisson equations, please refer, for instance, to Refs. 1–3, 14, and 15 and references therein.

Our aim is to show the global existence of weak solution to the free boundary problem (1.1)–(1.3) with initial data \(\rho_0 \in W^{1,2n}, \inf \rho_0 > 0\), and \(u_0 \in L^{2n}\), for some \(n \in \mathbb{N}\), satisfying \((2n - 1) / 2n > \theta\). In contrast to Ref. 5 the Sobolev exponent \(n\) in this paper is less [we just require \((2n - 1) / 2n > \theta\) for some \(n \in \mathbb{N}\)], and the examples of admissible viscosity coefficient include \(\mu(\rho) = \rho^\theta\) for \(0 < \theta < 1\) in Ref. 5 and \(\mu(\rho) = \mu_0 > 0\) for any \(\rho \geq 0\). Our ideas mainly come from Refs. 5 and 8.

To solve the free boundary value problem (1.1)–(1.3), it is convenient to convert the free boundaries to fixed boundaries by using Lagrangian coordinates. Let

\[y = \int_{a(t)}^{\xi} \rho(\xi, t) d\xi, \quad \tau = t.\]

Then the free boundaries \(x = a(t)\) and \(x = b(t)\) become \(y = 0\) and \(y = \int_{a(t)}^{b(t)} \rho(\xi, t) d\xi = \int_{a}^{b} \rho_{0}(\xi) d\xi = 1\) by the conservation of mass and assuming \(\int_{a}^{b} \rho_{0}(\xi) d\xi = 1\) without loss of generality.

Hence, in Lagrangian coordinates, the free boundary value problem (1.1)–(1.3) can be rewritten as follows:

\[\rho_{\tau} + \rho^2 u_{\tau} = 0,\]  
\[\rho(\tau, \cdot) = \rho_{0}(\xi), \quad a(t) = a_{0}, \quad b(t) = b_{0}.\]
\[ u_t + P_y + \rho \Phi_y = (\mu(\rho) \rho u)_y, \quad (1.5b) \]
\[ \rho(\rho \Phi)_y = 4 \pi g \left( \rho - \frac{1}{\int_0^1 \rho^{-1} dy} \right), \quad (y, \tau) \in (0, 1) \times (0, \infty), \quad (1.5c) \]

with initial data
\[ (\rho, u)|_{y=0} = (\rho_0(y), u_0(y)), \quad \text{on} \ 0 \leq y \leq 1, \quad (1.6) \]
and the boundary conditions
\[ (\mu(\rho) \rho u_y - P)|_{y=0} = 0, \quad \rho \Phi_y|_{y=0} = 0 \quad \text{for} \ \tau \geq 0. \quad (1.7) \]

**Notation:** Throughout this paper, \( L^p = L^p(0, 1) \) denotes usual Lebesgue space with the norm \( \| \cdot \|_p \) and \( W^{k,p} = W^{k,p}(0, 1) \) for the usual Sobolev space with the norm \( \| \cdot \|_{W^{k,p}} \) and \( C^{0,\beta} = C^{0,\beta}([0, 1] \times [0, T]) \) for the Banach space of functions on \([0, 1] \times [0, T]\) with the norm \( \| \cdot \|_{C^{0,\beta}} \) which are uniformly Hölder continuous with exponents \( \beta \) in \( x \) and \( \beta/2 \) in \( t \). \( H^k = W^{k,2}(0, 1), k \in \mathbb{N}. \)

We use \( C \) and \( C(T) \) to denote positive constants independent of \( T \) and depending on \( T \), respectively, but they both may depend on \( \|u_0\|_{L^2}, \|\rho_0\|_{W^{1,2}} \) and other known constants.

The main result can be stated as follows.

**Theorem 1.1:** Assume that the viscosity coefficient \( \mu(\rho) \) satisfies (1.4), the initial data \( \rho_0 \in W^{1,2n}, \inf_0 \rho_0 > 0 \), and \( u_0 \in L^{2n} \) for some \( n \in \mathbb{N} \) satisfying \( (2n-1)/2n > \theta \). Then there exists a global weak solution \((\rho, u, \Phi)\) to the fixed boundary value problem (1.6) and (1.7) such that for any \( T \in (0, \infty), \)

\[ \rho \in L^\infty(0, T; W^{1,2n}), \quad \rho \in L^2(0, T; L^2), \]
\[ u \in L^\infty(0, T; L^{2n}) \cap L^2(0, T; H^1), \quad \Phi \in L^\infty(0, T; W^{2,2n}), \quad \Phi_{y\tau} \in L^2(0, T; L^2), \]

Moreover
\[ \rho + \rho^2 u_y = 0, \quad \rho(y, 0) = \rho_0(y), \quad (\rho \Phi)_y = 4 \pi g \left( 1 - \frac{1}{\int_0^1 \rho^{-1} dy} \right), \quad \rho \Phi|_{y=0} = 0 \]
hold for a.e. \( y \in (0, 1) \) and \( \tau \in [0, T], \) and
\[ \int_0^T \int_0^1 \{ u \varphi + (P - \mu(\rho) \rho u)_y \varphi_y - \rho \Phi \varphi \} dy d\tau + \int_0^1 u_0(y) \varphi(y, 0) dy = 0 \]
for any \( \varphi \in C_0^\infty(Q) \), where \( Q = \{(y, \tau) | 0 \leq y \leq 1, 0 \leq \tau < T\}. \)

**Remark 1.2:** If in addition to the regularity assumptions about the initial data in Theorem 1.1, we suppose \( u_0 \in H^1 \), then by the similar method as that in Ref. 5 we can easily get
\[ \rho \in L^\infty(0, T; W^{1,2n}), \quad \rho \in L^2(0, T; L^2), \]
\[ u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \quad u \in L^2(0, T; L^2), \]
\[ \Phi \in L^\infty(0, T; W^{2,2n}) \],
\[ \rho(y, \tau) \geq C(T) > 0. \]

With the stronger regularity of the solution above, if \((\rho, u, \Phi)\) is a global weak solution to the fixed boundary value problem (1.5)–(1.7), then \((\rho, u, \Phi + h(\tau))\) is also a global weak solution to the problem (1.5)–(1.7). Therefore the uniqueness of the global weak solution referred above means in the sense of up to a function of \(\tau\) with respect to \(\Phi\).

**Remark 1.3:** In fact, by the similar method as that in Ref. 9 we can get the global existence of weak solution \((\rho, u, \Phi)\) to the fixed boundary value problem (1.5)–(1.7) for the case

\[ \mu(\rho) \geq c_0, \quad \forall \rho \geq 1, \]

\[ \mu(\rho) \geq c_0 \rho^\theta, \quad \forall 0 \leq \rho < 1 \quad \text{for } \theta = 1, \]

where \(c_0\) is a positive constant. But the initial assumptions are asked to be stronger than that in Theorem 1.1. For the details, please refer to Ref. 9 and references therein.

The rest of this paper is organized as follows. In Sec. II, we obtain a priori estimates under the assumptions that the solution of (1.5)–(1.7) is classical. In Sec. III, we obtain the global existence of the classical solution of (1.5)–(1.7) with smooth initial data \(\rho_0 \in C^{1+\beta}\) and \(u_0 \in C^{2+\beta}\). Then using the compactness theorems of Sobolev space, we get the existence of global weak solution.

II. A PRIORI ESTIMATES

In this section, we will give some a priori estimates for \((\rho, u, \Phi)\). Let \((\rho(y, \tau), u(y, \tau), \Phi(y, \tau))\) be a classical solution of (1.5)–(1.7) and \(\rho(y, \tau) > 0\) for all \((y, \tau) \in [0, 1] \times [0, T]\), where \(T\) is an any positive constant.

**Lemma 2.1:** (Basic energy estimates) Under the conditions of Theorem 1.1, the following energy estimates hold:

\[ \|\rho \Phi\|_{L^\infty([0,1] \times [0,T])} \leq C \]  \hspace{1cm} (2.1)

and

\[ \sup_{0 \leq \tau \leq T} \int_0^1 (u^2 + \rho^{-1}) dy + \int_0^T \int_0^1 \mu(\rho) \rho u^2 \, dy \, ds \leq C(T). \]  \hspace{1cm} (2.2)

**Proof:** Multiplying (1.5c) by \(\rho^{-1}\), integrating the resulting equation with respect to \(y\) over \((0, y)\) and using the boundary conditions (1.7), we have

\[ \rho \Phi_y = 4 \pi g \left( y - \frac{\int_0^y \rho^{-1}(z, \tau) \, dz}{\int_0^1 \rho^{-1}(z, \tau) \, dz} \right), \]  \hspace{1cm} (2.3)

which implies

\[ |\rho \Phi_y| \leq 4 \pi g \left( y + \frac{\int_0^y \rho^{-1}(z, \tau) \, dz}{\int_0^1 \rho^{-1}(z, \tau) \, dz} \right) \leq 8 \pi g. \]  \hspace{1cm} (2.4)

This shows that (2.1) holds.

Next, we turn to prove (2.2). Multiplying (1.5a) by \(\rho^{-2}\) and (1.5b) by \(u\), respectively, and summing them up, then integrating the resulting equation with respect to \((y, \tau)\) over \((0, 1) \times (0, \tau)\), using the boundary conditions (1.7), and integration by parts, we have
\[
\int_0^1 \left( \frac{1}{2} u^2 + \frac{1}{\gamma - 1} \rho \varphi \right) dy + \int_0^T \int_0^1 \mu(\rho) \rho u y \varphi dy ds + \int_0^T \int_0^1 \rho u \Psi_y dy ds = \int_0^1 \left( \frac{1}{2} u_0^2 + \frac{1}{\gamma - 1} \rho_0 \varphi \right) dy,
\]

which implies by (2.4) and Cauchy’s inequality

\[
\int_0^1 (u^2 + \rho^{-1}) dy + \int_0^T \int_0^1 \mu(\rho) \rho u y \varphi dy ds \leq C(T) + C \int_0^T \int_0^1 u^2 dy ds.
\]

By (2.6) and Gronwall’s inequality, we get (2.2) and the proof of Lemma 2.1 is completed.

**Lemma 2.2:** For any \((y, \tau) \in [0, 1] \times [0, T]\), we have

\[
\rho(y, \tau) \leq C(T).
\]

**Proof:** Denote

\[
M(\rho) = \int_1^\rho \frac{\mu(\xi)}{\xi} d\xi,
\]

where \(\mu(\rho)\) is defined by (1.4). Then \(M: (0, \infty) \rightarrow (-\infty, +\infty)\) is a strictly increasing map.

It follows from (1.5a) and (1.5b) that

\[
u_x + P_y + \rho \Phi_y + (M(\rho))_{y\tau} = 0.
\]

Integrating (2.7) over \([0, y] \times [0, \tau] \subset [0, 1] \times [0, T]\), and using the boundary conditions (1.7), we have

\[
M(\rho) + \int_0^\tau P(y, s) ds \leq M(\rho_0) + \int_0^1 (u^2 + u_0^2) dy + \int_0^\tau \left( 1 + \int_0^1 \rho \Phi_y^2 dy \right) ds \leq C(T).
\]

Hence

\[
\rho \leq M^{-1}(C(T)) \leq C(T),
\]

where \(M^{-1}(\cdot)\) denotes the inverse function of \(M(\rho)\).

This proves Lemma 2.2.

**Lemma 2.3:** Let the positive integer \(n\) satisfy \((2n - 1)/2n > \theta\) and \(0 < \theta < 1\). Then

\[
\int_0^1 \left( \rho \Phi_{yy}^2 + u^2 \right) dy + \int_0^T \int_0^1 \mu(\rho) \rho u y \varphi dy ds \leq C(T) \quad \text{for any } \tau \in [0, T].
\]

**Proof:** Multiplying (1.5b) by \(u^{2n-1}\), then integrating it with respect to \(y\) over \((0,1)\), integrating by parts, and using the boundary conditions (1.7), we have

\[
\frac{1}{2n} \frac{d}{d \tau} \int_0^1 u^{2n} dy + (2n - 1) \int_0^1 \mu(\rho) \rho u^{2n-2} u_y^2 dy - (2n - 1) \int_0^1 P u^{2n-2} u_y dy + \int_0^1 \rho \Phi_y u^{2n-1} dy = 0.
\]

By Young’s inequality \(ab \leq (1/p)a^p + (1/q)b^q\), where \(a, b > 0\), \(1/p, q < \infty\), \((1/p) + (1/q) = 1\), we have
\[ \frac{1}{2n} \frac{d}{d\tau} \int_0^1 u^{2n} dy + (2n-1) \int_0^1 \mu(\rho) \rho u^{2n-2} u'_y dy = \]

\[ = (2n-1) \int_0^1 \rho u^{2n-1} u_y (\mu(\rho)\rho)^{\frac{1}{2}} (\mu(\rho)\rho)^{\frac{1}{2}} dy - \int_0^1 \rho \Phi_y u^{2n-1} dy \leq C \int_0^1 u^{2n} dy \]

\[ + C \int_0^1 \rho^2 \Phi_y^2 dy + \frac{2n-1}{2} \int_0^1 \mu(\rho) \rho u^{2n-2} u''_y dy + C \int_0^1 P^2(\mu(\rho)\rho)^{-1} u^{2n-2} dy \leq C \int_0^1 u^{2n} dy \]

\[ + C \int_0^1 \rho^2 \Phi_y^2 dy + \frac{2n-1}{2} \int_0^1 \mu(\rho) \rho u^{2n-2} u''_y dy + C \int_0^1 P^2(\mu(\rho)\rho)^{-n} dy, \]

which implies

\[ \frac{1}{2n} \frac{d}{d\tau} \int_0^1 u^{2n} dy + \frac{2n-1}{2} \int_0^1 \mu(\rho) \rho u^{2n-2} u''_y dy \leq C \int_0^1 u^{2n} dy + C \int_0^1 \rho^2 \Phi_y^2 dy \]

\[ + C \int_0^1 \rho^2 \gamma^{-\eta} u^{2n-\eta} (\rho^{\eta}) dy. \quad (2.8) \]

Now we estimate the third term on the right hand side of (2.8) as follows.

We have by (1.4) and Lemma 2.2

\[ \int_0^1 \rho^{2n-\gamma} \mu^{-\eta}(\rho) dy = \int_{[\gamma \in [0,1] | \rho(y,\gamma) \geq 1]} \rho^{2n-\gamma} \mu^{-\eta}(\rho) dy + \int_{[\gamma \in [0,1] | \rho(y,\gamma) < 1]} \rho^{2n-\gamma} \mu^{-\eta}(\rho) dy \]

\[ \leq C(T) + c_0 \int_{[\gamma \in [0,1] | \rho(y,\gamma) < 1]} \rho^{2n-\gamma} \mu^{-\eta} dy \leq C(T). \quad (2.9) \]

Substituting (2.1) and (2.9) into (2.8), we have

\[ \frac{d}{d\tau} \int_0^1 u^{2n} dy + \frac{2n-1}{2} \int_0^1 \mu(\rho) \rho u^{2n-2} u''_y dy \leq C \int_0^1 u^{2n} dy + C(T). \quad (2.10) \]

Applying Gronwall’s inequality to (2.10) and using (2.1), we obtain

\[ \int_0^1 \rho^2 \Phi_y^2 dy + \int_0^1 u^{2n} dy + \int_0^\tau \int_0^1 \mu(\rho) \rho u^{2n-2} u''_y dy ds \leq C(T) \]

for any \( \tau \in [0,T] \).

The proof of Lemma 2.3 is completed.

Lemma 2.4: For \( n \in N \) with \( (2n-1)/2n > \theta \) and \( 0 < \theta < 1 \), we have

\[ \int_0^1 \mu^{2n}(\rho) \rho^{2n} \Phi_y^2 dy \leq C(T) \quad \text{for any} \quad \tau \in [0,T]. \]

Proof: Integrating (2.7) from 0 to \( \tau \), we have

\[ u - u_0 + \int_0^\tau (P_y + \rho \Phi_y) ds + \mu(\rho) \rho^{-1} \rho_y - \mu(\rho_0) \rho_0^{-1} \rho_0 y = 0. \quad (2.11) \]

Multiplying (2.11) by \( (\mu(\rho) \rho^{-1} \rho_y)^{2n-1} \) and integrating the resulting equation with respect to \( y \) over \((0,1)\), we obtain
\[ \int_0^1 (\mu(\rho)\rho^{-1}\rho_y)^{2\gamma} dy = -\int_0^1 (u - u_0)(\mu(\rho)\rho^{-1}\rho_y)^{2\gamma - 1} dy - \int_0^1 (\mu(\rho)\rho^{-1}\rho_y)^{2\gamma - 1} \int_0^\tau (P_y + \rho\Phi_y) ds dy \]
\[ + \int_0^1 (\mu(\rho_0)\rho_0^{-1}\rho_0\rho_y)(\mu(\rho)\rho^{-1}\rho_y)^{2\gamma - 1} dy \leq \frac{1}{2} \int_0^1 (\mu(\rho)\rho^{-1}\rho_y)^{2\gamma} dy + C \int_0^1 u^{2\gamma} dy \]
\[ + C \int_0^1 u_0^{2\gamma} dy + C(T) \int_0^\tau \int_0^1 \rho_0^{2\gamma} dy ds + C(T) \int_0^\tau \int_0^1 \rho_0^{2\gamma} dy ds \]
\[ + C \int_0^1 (\mu(\rho_0)\rho_0^{-1}\rho_0\rho_y)^{2\gamma} dy. \]

Hence
\[ \frac{1}{2} \int_0^1 (\mu(\rho)\rho^{-1}\rho_y)^{2\gamma} dy \leq C(T) \int_0^\tau \int_0^1 \rho^{2\gamma(\gamma - 1)} \rho_y^{2\gamma} dy ds \]
\[ = C(T) \int_0^\tau \int_0^1 \rho^{2\gamma} \mu^{-2\gamma}(\rho) \mu \rho^{2\gamma} \rho_y^{2\gamma} dy ds. \quad (2.12) \]

Now we claim for any \( \rho \geq 0, \rho^{2\gamma}\mu^{-2\gamma}(\rho) \leq C(T). \) In fact, when \( 0 \leq \rho < 1, \mu(\rho) \approx c_0\rho^\theta, \) we have
\[ \rho^{2\gamma}\mu^{-2\gamma}(\rho) \leq \left( \frac{1}{c_0} \rho^{-\theta} \right)^{2n} \leq C(T), \]
and when \( \rho \geq 1, \mu(\rho) \approx c_0, \) we have
\[ \rho^{2\gamma}\mu^{-2\gamma}(\rho) \leq C(T). \]

By Gronwall’s inequality, we obtain from (2.12) that
\[ \int_0^1 (\mu(\rho)\rho^{-1}\rho_y)^{2\gamma} dy \leq C(T). \]

This proves Lemma 2.4.

Now we give the boundary estimates for the density function \( \rho(y, \tau), \) which will frequently be used later.

**Lemma 2.5:** Under the conditions of Theorem 1.1, we have
\[ \rho(h, \tau) \geq C(T) > 0 \quad \text{for } h = 0, 1, \tau \in [0, T]. \]

**Proof:** By (1.7), we have
\[ \mu(\rho)\rho u_y = \rho^\gamma \quad \text{at } y = h. \]
Combining (1.5a), we have
\[ \mu(\rho)\rho^{-1}\rho_x + \rho^\gamma = 0 \quad \text{at } x = h. \quad (2.13) \]
Multiplying (2.13) by \( (\theta - \gamma)\mu^{-1}(\rho)\rho^{\theta - \gamma}, \) we obtain
\[ (\rho^{\theta - \gamma})_x = (\gamma - \theta) \frac{\rho^\theta}{\mu(\rho)} \quad \text{at } x = h. \quad (2.14) \]
Integrating (2.14) with respect to \( \tau \) over \((0, \tau),\) we get
\[
\rho^{\alpha-\gamma} = \int_0^\tau (\gamma - \theta) \frac{\rho^\theta}{\mu(\rho)} ds + \rho_0^{\alpha-\gamma} \leq (\gamma - \theta) \int_0^\tau \frac{\rho^\theta}{\mu(\rho)} ds + \rho_0^{\alpha-\gamma} \quad \text{at } x = h.
\]
(2.15)

By the similar discussion of Lemma 2.4, we obtain from (1.4) that
\[
\frac{\rho^\theta}{\mu(\rho)} \leq C(T) \quad \text{for any } \rho \geq 0.
\]
(2.16)

Substituting (2.16) into (2.15) and using \(0 < \theta < 1 < \gamma\), we have
\[
\rho(h, \tau) \geq C(T) > 0 \quad \text{for any } \tau \in [0, T].
\]

This proves Lemma 2.5.

**Lemma 2.6:** Under the conditions of Theorem 1.1, we have
\[
\int_0^\tau \|\rho^{\theta+1}u(\cdot, s)\|_{L^\infty([0, 1])} ds \leq C(T).
\]

**Proof:** By Sobolev embedding theorem \(W^{1,1}([0, 1]) \to L^\infty([0, 1])\), we have
\[
\int_0^\tau \|\rho^{\theta+1}u(\cdot, s)\|_{L^\infty([0, 1])} ds \leq \int_0^\tau \int_0^1 \rho^{\theta+1}|u| dy ds + \int_0^\tau \int_0^1 (\rho^{\theta+1}|u_y|) dy ds \leq C \int_0^\tau \int_0^1 (\rho^{2\theta+2} + |u|^2) dy ds
\]
\[
+ C \int_0^\tau \int_0^1 \rho^{\theta+1}u_y dy ds + \int_0^\tau \int_0^1 (\theta + 1)\rho^\theta \mu u dy ds \leq C(T)
\]
\[
+ C \int_0^\tau \int_0^1 \rho^{\theta+1}u^2 dy ds + C \int_0^\tau \int_0^1 \rho^{\theta+1}u_y dy ds + C \int_0^\tau \int_0^1 \rho^{2\theta} \mu^2 u_y^2 dy ds
\]
\[
+ C \int_0^\tau \int_0^1 u^2 dy ds \leq C(T) + C \int_0^\tau \int_{y \in [0, 1]} \rho^{\theta+1}u^2 dy ds + C \int_0^\tau \int_{y \in [0, 1]} \rho^{2\theta} \mu^2 u_y^2 dy ds
\]
\[
+ C \int_0^\tau \int_{y \in [0, 1]} \rho^{2\theta} \mu u_y^2 dy ds \leq C(T)
\]
\[
+ C \int_0^\tau \int_{y \in [0, 1]} \frac{\rho^\theta}{\mu(\rho)} u^2 dy ds
\]
\[
+ C \int_0^\tau \int_{y \in [0, 1]} \mu(\rho) u^2 dy ds
\]
\[
+ C \int_0^\tau \int_{y \in [0, 1]} \rho^{2\theta} \mu^2 (\rho) u^2 dy ds \leq C(T)
\]
\[
+ C \int_0^\tau \int_{y \in [0, 1]} \rho^2 \mu^2 (\rho) u^2 dy ds \leq C(T)
\]
\[
+ C \int_0^\tau \int_{y \in [0, 1]} \mu(\rho) u^2 dy ds + C \int_0^\tau \int_{y \in [0, 1]} \mu^2 (\rho) u^2 dy ds \leq C(T).
\]

Here we have used (1.4), Lemma 2.1, and Lemma 2.4 with \(n = 1\).

This proves Lemma 2.6.
The next lemma plays an important role to obtain the positive lower bound of the density function $\rho(y, \tau)$.

**Lemma 2.7:** If the conditions of Theorem 1.1 are satisfied, then
\[
\int_0^1 \frac{1}{\rho(y, \tau)} dy \leq C(T) \quad \text{for any } \tau \in [0, T].
\]

**Proof:** By (1.5a), we have
\[
d \int_0^1 \frac{1}{\rho} dy = - \int_0^1 \frac{\rho_{y}}{\rho^2} dy = \int_0^1 u dy = u(1, \tau) - u(0, \tau).
\]

Integrating the above equality with respect to $\tau$ over $(0, \tau)$, we have by Lemma 2.5 and Lemma 2.6 that
\[
\int_0^1 \frac{1}{\rho} dy = \int_0^1 \frac{1}{\rho} dy + \int_0^\tau [u(1, s) - u(0, s)] ds \leq C + \int_0^\tau \left\{ \frac{\rho^{\alpha_1}(1, s)u(1, s)}{\rho^{\alpha_1}(1, s)} - \frac{\rho^{\alpha_1}(0, s)u(0, s)}{\rho^{\alpha_1}(0, s)} \right\} ds
\leq C + C(T) \int_0^\tau (|\rho^{\alpha_1}(1, s)|u(1, s)| + |\rho^{\alpha_1}(0, s)|u(0, s)|) ds \leq C
\]
\[
+ C(T) \int_0^\tau \|\rho^{\alpha_1}u(\cdot, s)\|_{L^\infty([0, 1])} ds \leq C(T).
\]

This proves Lemma 2.7.

Base on the above lemma, we can obtain the positive lower bound of the density function $\rho(y, \tau)$. With this crucial estimate on the positive lower bound for the density function, we can study the other properties of the solution $(\rho, u, \Phi)$.

**Lemma 2.8:** Under the conditions of Theorem 1.1, we have
\[
\rho(y, \tau) \equiv C(T) > 0 \quad \text{for any } (y, \tau) \in [0, 1] \times [0, T].
\]

**Proof:** By Sobolev embedding theorem $W^{1,1}([0, 1]) \to L^\infty([0, 1])$, we have by Lemma 2.4 and Lemma 2.7
\[
\frac{1}{\rho(y, \tau)} \leq \int_0^1 \frac{1}{\rho(y, \tau)} dy + \int_0^1 \frac{|\rho_{y}|}{\rho} dy \leq C(T) + \int_0^1 \left\{ \frac{|\mu(\rho)|^{\rho^{-1} - 1} \rho}{\rho^\mu} \right\} dy \equiv C(T)
\]
\[
+ \left( \int_0^1 \frac{1}{\rho} dy \right)^{1/2n} \left( \int_0^1 \frac{1}{\rho^\mu} dy \right)^{1/2n} \left( \int_0^1 \frac{1}{\rho^2} dy \right)^{(2n-1)/2n} \equiv C(T) + C(T)
\]
\[
\times \left( \int_0^1 \frac{1}{\rho^2} dy \right)^{(2n-1)/2n} \equiv C(T) + C(T) \max_{y \in [0, 1]} \left( \frac{1}{\rho^{1/2n} \mu(\rho)} \right).
\]

Now we estimate the second term on the right-hand side of (2.17). By (1.4), we have for any $\rho \geq 0$
\[
\rho^{-1/2n} \mu^{-1}(\rho) \leq \frac{1}{c_0} (\rho^{-1/2n - \theta} + 1).
\]

Substituting (2.18) into (2.17), we have
\[
\frac{1}{\rho(y, \tau)} \leq C(T) + C(T) \max_{y \in [0,1]} \rho^{-1(1/2n+\theta)}. \tag{2.19}
\]

Since \( \theta < (2n-1)/2n, 1/2n + \theta < 1 \), it follows from (2.19) and Young’s inequality that

\[
\max_{y \in [0,1]} \rho^{-1}(y, \tau) \leq C(T) + \frac{1}{2} \max_{y \in [0,1]} \rho^{-1}(y, \tau),
\]

which implies

\[
\max_{y \in [0,1]} \rho^{-1}(y, \tau) \leq C(T),
\]

i.e.,

\[
\rho(y, \tau) \geq C(T) > 0.
\]

This proves Lemma 2.8.

**Lemma 2.9:** Under the conditions of Theorem 1.1, we have

\[
\int_0^T \| \rho(t, \cdot) \|^2_{L^2} d\tau \leq C(T) \tag{2.20}
\]

and

\[
\int_0^T \| \Phi(t, \cdot) \|^2_{L^2} d\tau \leq C(T). \tag{2.21}
\]

**Proof:** (2.20) is a consequence of Eq. (1.5a), Lemma 2.1, Lemma 2.2, and Lemma 2.8.

Next, we turn to prove (2.21). From (2.3), we have

\[
\Phi_y = 4\pi g \frac{1}{\rho} \left( y - \int_0^y \frac{\rho}{\rho'} dz \right), \tag{2.22}
\]

which implies

\[
\Phi_{yy} = -4\pi g \rho^{-2} \left( y - \int_0^y \frac{\rho}{\rho'} dz \right) - 4\pi g \rho^{-1} \left( \int_0^y \frac{\rho}{\rho'} dz \right)^2 dy.
\]

We have from (2.23), Lemma 2.2, Lemma 2.8, and (2.20) that

\[
\int_0^T \| \Phi_{yy}(\cdot, \tau) \|^2_{L^2} d\tau \leq C(T) \int_0^T \| \rho(t, \cdot) \|^2_{L^2} d\tau + C(T) \int_0^T \left( \int_0^1 \frac{\rho}{\rho'} dz \right)^2 dy d\tau
\]

\[
+ C(T) \int_0^T \left( \int_0^1 \frac{\rho}{\rho'} dz \right)^2 dy d\tau \leq C(T),
\]

which implies (2.21). And we complete the proof of Lemma 2.9.
III. PROOF OF THEOREM 1.1

In this section, we use \( C(T) \) to denote positive constant depending on \( T, \|u_0\|_{L^2}, \|\rho_0\|_{C^{1+\beta}}, \) and other known constants.

Before proving Theorem 1.1, we need the following lemmas.

**Lemma 3.1**: Assume the initial data \( \rho_0 \in C^{1+\beta}([0,1]), u_0 \in C^2([0,1]) \) for some \( 0 < \beta < 1 \).

Then there exists a global classical solution \( (\rho, u, \Phi) \) to (1.6) and (1.7) such that for any \( T > 0, \)

\[
(\rho, \rho_\gamma, \rho_{\gamma\gamma}, u, u_y, u_{yy}, \Phi, \Phi_y, \Phi_{yy}) \in (C^{\beta,1}(Q_T))^{11},
\]

where \( Q_T = [0,1] \times [0,T] \).

We need the following estimates to prove Lemma 3.1.

**Lemma 3.2**: Under the conditions of Lemma 3.1, we have

\[
\sup_{\tau \in [0,T]} \|u(\cdot, \tau)\|_{H^1} + \int_0^T \|u(\cdot, s)\|_{L^2} ds \leq C(T).
\]

**Proof**: Denote \( \bar{\mu}(\rho) = \mu(\rho) \rho \). Multiplying (1.5b) by \( u \), and integrating the resulting equation with respect to \( y \) over \( (0,1) \), we have

\[
\int_0^1 u_y^2 dy + \int_0^1 (\rho \Phi_y + P_y)u_y dy - \int_0^1 (\bar{\mu}(\rho_{yy}) + \rho_{yy})u_y dy = 0.
\]

Integrating by parts and using boundary conditions (1.7), we have

\[
\int_0^1 u_y^2 dy + \frac{1}{2} \int_0^1 \frac{d}{d\tau} \bar{\mu}(\rho) u_y^2 dy - \int_0^1 \frac{d}{d\tau} P_y u_y dy + \int_0^1 P''(\rho) \rho_{yy} u_y dy - \frac{1}{2} \int_0^1 \bar{\mu}'(\rho) \rho_{yy} u_y dy + \int_0^1 \rho \Phi_y u_y dy
\]

\[
= 0.
\]

Utilizing (1.5a) and (1.7), we get

\[
\int_0^1 u_y^2 dy + \frac{1}{2} \int_0^1 \frac{d}{d\tau} \bar{\mu}(\rho) u_y^2 dy - \int_0^1 \frac{d}{d\tau} P_y u_y dy = -\frac{1}{2} \int_0^1 \bar{\mu}'(\rho) \rho^2 u^2_y dy + \gamma \int_0^1 \rho^{\gamma-1} u^2_y dy - \int_0^1 \rho \Phi_y u_y dy
\]

\[
\leq C(T) \int_0^1 \bar{\mu}(\rho) |u^3_y| dy + C(T) \int_0^1 \bar{\mu}(\rho) u^2_y dy
\]

\[
+ \frac{1}{2} \int_0^1 u_y^2 dy + \frac{1}{2} \int_0^1 \rho^2 \Phi_y^2 dy,
\]

which implies by Lemma 2.1

\[
\frac{1}{2} \int_0^1 u_y^2 dy + \frac{1}{2} \int_0^1 \frac{d}{d\tau} \bar{\mu}(\rho) u_y^2 dy - \int_0^1 \frac{d}{d\tau} P_y u_y dy \leq C(T) \int_0^1 \bar{\mu}(\rho) u^2_y dy + C(T) \int_0^1 \|u(\cdot, \tau)\|_{L^2} \int_0^1 \bar{\mu}(\rho) u^2_y dy + C(T).
\]

By Sobolev embedding theorem \( W^{1,1}([0,1]) \hookrightarrow L^\infty([0,1]) \) and Lemma 2.2, Lemma 2.8, we have
\[ \| u_{\gamma}(\cdot, \tau) \|_{L^2} \leq C(T) \| (\bar{\mu}(\rho)u_{\gamma} - P)(\cdot, \tau) \|_{L^2} + C(T) \leq C(T) \| (\bar{\mu}(\rho)u_{\gamma} - P)(\cdot, \tau) \|_{L^1} + C(T) \| (\bar{\mu}(\rho)u_{\gamma} - P)(\cdot, \tau) \|_{L^1} + C(T) \]

By using boundary conditions (1.7) and (2.3), Lemma 2.4, Lemma 2.7, and Lemma 2.8, we get

\[ \frac{1}{2} \int_0^1 u_{\gamma}^2 dy + \frac{1}{2} \int_0^1 \bar{\mu}(\rho)u_{\gamma}^2 dy - \int_0^1 \bar{\mu}(\rho)u_{\gamma}^2 dy \leq C(T) \left( \int_0^1 \bar{\mu}(\rho)u_{\gamma}^2 dy \right) + C(T) \]

which implies

\[ \frac{1}{4} \int_0^1 u_{\gamma}^2 dy + \frac{1}{2} \int_0^1 \bar{\mu}(\rho)u_{\gamma}^2 dy - \int_0^1 \bar{\mu}(\rho)u_{\gamma}^2 dy \leq C(T) \left( \int_0^1 \bar{\mu}(\rho)u_{\gamma}^2 dy \right) + C(T). \] (3.5)

By Gronwall’s inequality and Lemma 2.1, we obtain

\[ \int_0^T \int_0^1 u_{\gamma}^2 ds + \int_0^1 u_{\gamma}^2 dy \leq C(T). \] (3.6)

By (3.6) and Lemma 2.1, the proof of Lemma 3.2 is completed.

**Lemma 3.3:** Under the conditions of Lemma 3.1, we have

\[ \sup_{\tau \in [0,T]} \| u_{\gamma}(\cdot, \tau) \|_{L^2} + \int_0^T \| u_{\gamma}(\cdot, s) \|^2_{L^2} ds \leq C(T). \]

**Proof:** Differentiating (1.5b) with respect to \( \tau \), multiplying the resulting equation by \( u_{\gamma} \), and then integrating it with respect to \( y \) over \((0,1)\), we get

\[ \frac{1}{2} \int_0^1 u_{\gamma}^2 dy + \int_0^1 u_{\gamma}P_{\gamma} dy + \int_0^1 (\rho \Phi_{\gamma}) u_{\gamma} dy - \int_0^1 (\bar{\mu}(\rho)u_{\gamma})_{\gamma} u_{\gamma} dy = 0. \]

By using boundary conditions (1.7) and (2.3), Lemma 2.4, Lemma 2.7, and Lemma 2.8, we get
It follows from Lemma 3.2 that

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 u_\tau^2 dy + \int_0^1 \bar{\mu}(\rho)u_\tau^2 dy = \int_0^1 u_{\tau\tau} \rho dy + 4 \pi g \int_0^1 \left( \int_0^y \rho^{-1} dz \right) dy - \int_0^1 \bar{\mu}'(\rho) \rho u_\tau u_\tau dy
\]

\[
= \int_0^1 \bar{\mu}'(\rho) \rho^2 u_\tau^2 dy - \gamma \int_0^1 u_{\tau\tau} \rho \gamma u_\tau dy
\]

\[
+ 4 \pi g \int_0^1 u_{\tau\tau} \rho^2 - \int_0^1 \rho^2 \rho dz \int_0^1 \rho^{-1} dz + \int_0^1 \rho^2 \rho dz \int_0^1 \rho^{-1} dz
\]

\[
\leq \frac{1}{2} \int_0^1 \bar{\mu}(\rho) u_\tau^2 dy + C(T) \int_0^1 u_\tau^4 dy + \bar{C}(T) + C(T) \int_0^1 |u_\tau| dy
\]

\[
\times \|u(\cdot, \tau)\|_{L^\infty([0,1])} dy \leq \frac{1}{2} \int_0^1 \bar{\mu}(\rho) u_\tau^2 dy + C(T) \int_0^1 u_\tau^4 dy + \bar{C}(T)
\]

\[
+ C(T) \int_0^1 u_\tau^2 dy + C(\cdot, \tau) \|u(\cdot, \tau)\|_{L^2([0,1])}^2.
\]

(3.7)

It follows from Lemma 3.2 that

\[
\|u(\cdot, \tau)\|_{L^\infty([0,1])} \leq C \sup_{\tau \in [0,T]} \|u(\cdot, \tau)\|_{H^2} \leq \bar{C}(T),
\]

(3.8)

and by (3.4), Lemma 2.2, Lemma 2.8, and Lemma 3.2, we have

\[
\int_0^1 u_\tau^4 dy \leq \|u_\tau(\cdot, \tau)\|_{L^2}^2 \int_0^1 u_\tau^2 dy \leq \bar{C}(T) + \bar{C}(T) \int_0^1 u_\tau^2 dy.
\]

(3.9)

Substituting (3.8) and (3.9) into (3.7), we deduce

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 u_\tau^2 dy + \frac{1}{2} \int_0^1 \bar{\mu}(\rho) u_\tau^2 dy \leq \bar{C}(T) + \bar{C}(T) \int_0^1 u_\tau^2 dy.
\]

By Gronwall’s inequality, we have

\[
\int_0^1 u_\tau^2 dy + \int_0^T \int_0^1 u_\tau^2 dy ds \leq \bar{C}(T).
\]

The proof of Lemma 3.3 is completed.

By Lemma 3.2, Lemma 3.3, and Eq. (1.5b), we can get the following result.

**Corollary 3.4:** Under the conditions of Lemma 3.1, we have

\[
\|u(\cdot)\|_{L^\infty} \leq \bar{C}(T) \quad \text{for any } \tau \in [0,T].
\]

**Proof of Lemma 3.1:** The proof of Lemma 3.1 is based on a priori estimates (3.10) below that can be used to continue a local solution globally in time. The existence of a local solution can be obtained by linearization of Eq. (1.5), and using the Banach contraction theorem (cf. Refs. 6 and 11). So to complete the proof of Lemma 3.1, it suffices to establish the following a priori estimates:
\[
\|(\rho, \rho_t, \rho_y, \rho_{yy})\|_{C^{1,\beta/2}} + \|(u, u_y, u_{yy}, u_t)\|_{C^{1,\beta/2}} + \|(\Phi, \Phi_y, \Phi_{yy})\|_{C^{1,\beta/2}} \leq \tilde{C}(T). \tag{3.10}
\]

In fact, let \((\rho, u, \Phi)\) be a classical solution of (1.5)–(1.7) with initial data \(\rho_0 \in C^{1+\beta}([0,1]), u_0 \in C^2([0,1])\) Then, by Lemmas 2.1–2.8, Lemma 3.2, Lemma 3.3, and Corollary 3.4, we have for any \(\tau_1, \tau_2, \tau \in [0,T], y_1, y_2, y \in [0,1]\)

\[
|\rho(y, \tau_1) - \rho(y, \tau_2)| = \left| \int_{\tau_2}^{\tau_1} \rho'(y, s) ds \right| = \left| \int_{\tau_2}^{\tau_1} \rho'^2 u_y(y, s) ds \right| \leq \|\rho'^2 u_y\|_{L^\infty} |\tau_1 - \tau_2| \leq \tilde{C}(T) |\tau_1 - \tau_2|,
\]

and

\[
|\rho(y_1, \tau) - \rho(y_2, \tau)| = \left| \int_{y_2}^{y_1} \rho_y(z, \tau) dz \right| \leq |y_1 - y_2|^{1/2} \left( \int_0^1 \rho_y^2(z, \tau) dz \right)^{1/2} \leq C(T) |y_1 - y_2|^{1/2},
\]

which implies

\[
\|\rho\|_{C^{1/2,1/4}} \leq \tilde{C}(T). \tag{3.11}
\]

Similarly, we obtain

\[
\|u\|_{C^{1/2,1/4}} \leq \tilde{C}(T). \tag{3.12}
\]

By (2.3) and Lemma 2.2, Lemma 2.8, we have

\[
\|\Phi_y\|_{C^{1/2,1/4}} \leq \tilde{C}(T). \tag{3.13}
\]

It follows from (2.7) that

\[
(M(\rho)_y)_z = -u_z - \frac{\gamma \rho_y}{\mu(\rho)} M(\rho)_y - \rho \Phi_y.
\]

By Gronwall’s inequality and (3.11)–(3.13), we get

\[
\|M(\rho)_y\|_{L^\infty} \leq \tilde{C}(T) \tag{3.14}
\]

and

\[
\|M(\rho)_y\|_{C^{1/2,1/4}} \leq \tilde{C}(T). \tag{3.15}
\]

Combining (3.11) and (3.15), we deduce

\[
\|\rho_t\|_{C^{1/2,1/4}} \leq \tilde{C}(T). \tag{3.16}
\]

It follows from (3.11), (3.13), (3.16), and (1.5b) and Schauder theory of linear parabolic equations that

\[
\|(u, u_y, u_{yy}, u_t)\|_{C^{\alpha,\alpha/2}} \leq \tilde{C}(T),
\]

where \(\alpha = \min\{\beta, \frac{1}{2}\}\).

From above, we get

\[
\|(\rho, \rho_t, \rho_y, \rho_{yy})\|_{C^{\alpha,\alpha/2}} + \|(u, u_y, u_{yy}, u_t)\|_{C^{\alpha,\alpha/2}} + \|(\Phi, \Phi_y, \Phi_{yy})\|_{C^{\alpha,\alpha/2}} \leq \tilde{C}(T).
\]

Hence

\[
\|u\|_{C^{1,1/2}} + \|\rho\|_{C^{1,1/2}} + \|\Phi\|_{C^{1,1/2}} \leq \tilde{C}(T).
\]
Repeating the above arguments, we can get
\[ \| (u, u_x, u_{yy}, u_{tx}) \|_{C^{0,\alpha}} \leq \tilde{C}(T), \]
where
\[ \alpha = \min\{1, \beta\} = \beta, \]
which implies (3.10) and the proof of the Lemma 3.1 is completed.

**Lemma 3.5:** (Reference 10) Assume \( X \subset E \subset Y \) are Banach spaces and \( X \hookrightarrow E \). Then the following imbedding are compact:

(i) \( \varphi : \varphi \in L^q(0, T; X), \frac{\partial \varphi}{\partial t} \in L^1(0, T; Y) \hookrightarrow L^q(0, T; E), \quad \text{if} \quad 1 \leq q \leq \infty, \)

(ii) \( \varphi : \varphi \in L^r(0, T; X), \frac{\partial \varphi}{\partial t} \in L^r(0, T; Y) \hookrightarrow C([0, T]; E), \quad \text{if} \quad 1 < r \leq \infty. \)

**Proof of Theorem 1.1:** Mollifying the initial data like Ref. 5, we obtain
\[ \rho_0^\varepsilon \in C^{1+\beta}([0, 1]), \quad u_0^\varepsilon \in C^{2+\beta}([0, 1]) \]
for any \( \beta \in (0, 1) \), such that
\[ \rho_0^\varepsilon \to \rho_0, \quad \text{in} \quad W^{1,2n}, \quad u_0^\varepsilon \to u_0, \quad \text{in} \quad L^{2n}, \quad \text{as} \quad \varepsilon \to 0. \]

Consider the problem (1.5)–(1.7) with the initial data \( (\rho_0, u_0) \) replaced by \( (\rho_0^\varepsilon, u_0^\varepsilon) \). It follows from Lemma 3.1 that there exists a classical solution \( (\rho^\varepsilon, u^\varepsilon, \Phi^\varepsilon) \).

We obtain from Lemmas 2.1–2.9 that
\[ \sup_{0 \leq \tau \leq T} \| \rho^\varepsilon(\cdot, \tau) \|_{W^{1,2n}} \leq C(T), \quad \int_0^T \| \rho^\varepsilon(\cdot, s) \|_{L^2}^2 ds \leq C(T), \quad \rho^\varepsilon(\cdot, \tau) \geq C(T) > 0, \]
\[ \sup_{0 \leq \tau \leq T} \| \Phi^\varepsilon(\cdot, \tau) \|_{W^{1,2n}} \leq C(T), \quad \int_0^T \| \Phi^\varepsilon_{\tau}(\cdot, \tau) \|_{L^2}^2 d\tau \leq C(T), \]
\[ \sup_{0 \leq \tau \leq T} \| u^\varepsilon(\cdot, \tau) \|_{L^{2n}} \leq C(T), \quad \int_0^T \| u^\varepsilon(\cdot, s) \|_{L^4}^2 ds \leq C(T). \]

By the compactness theorem of Sobolev space, there exists a subsequence of \( \{(\rho^\varepsilon, u^\varepsilon, \Phi^\varepsilon)\} \), still denote by \( \{(\rho^\varepsilon, u^\varepsilon, \Phi^\varepsilon)\} \), such that as \( \varepsilon \to 0 \)
\[ \rho^\varepsilon \to \rho, \quad \text{weak}^* \quad \text{in} \quad L^\infty(0, T; L^{2n}), \]
\[ \rho^\varepsilon_{\tau} \to \rho_{\tau}, \quad \text{weak}^* \quad \text{in} \quad L^\infty(0, T; L^{2n}), \]
\[ \rho^\varepsilon \to \rho_{\tau}, \quad \text{weakly} \quad \text{in} \quad L^2(0, T; L^2), \]
\[ u^\varepsilon \to u, \quad \text{weak}^* \quad \text{in} \quad L^\infty(0, T; L^{2n}), \]
\[ u^\varepsilon_{\tau} \to u_{\tau}, \quad \text{weakly} \quad \text{in} \quad L^2(0, T; L^2), \]
\[ \Phi^e \to \Phi, \quad \text{weak} - * \text{ in } L^\infty(0,T; L^2), \]
\[ \Phi^e_y \to \Phi_y, \quad \text{weak} - * \text{ in } L^\infty(0,T; L^2), \]
\[ \Phi^e_{yy} \to \Phi_{yy}, \quad \text{weak} - * \text{ in } L^\infty(0,T; L^2). \]

Since \( \rho^e \) is bounded in \( L^\infty(0,T; W^{1,2}) \) and \( \rho^e_T \) is bounded in \( L^2(0,T; L^2) \), we obtain from Lemma 3.5 that
\[ \rho^e \to \rho \text{ in } C([0,T] \times [0,1]). \]

From above, the existence of the global weak solution is easily obtained.

By lower semicontinuities and the strong convergence of \( \rho^e \) in \( C([0,1] \times [0,T]) \), we have
\[
\sup_{0 \leq \tau \leq T} \| \rho(\cdot, \tau) \|_{W^{1,2}} \leq C(T), \quad \int_0^T \| \rho(\cdot, s) \|_{L^2}^2 ds \leq C(T), \quad \rho(y, \tau) \geq C(T) > 0,
\]
\[
\sup_{0 \leq \tau \leq T} \| \Phi(\cdot, \tau) \|_{W^{2,2}} \leq C(T), \quad \int_0^T \| \Phi_y(\cdot, \tau) \|_{L^2}^2 d\tau \leq C(T),
\]
\[
\sup_{0 \leq \tau \leq T} \| u(\cdot, \tau) \|_{L^2} \leq C(T), \quad \int_0^T \| u(\cdot, s) \|_{H^1}^2 ds \leq C(T).
\]

The proof of Theorem 1.1 is completed.

ACKNOWLEDGMENTS

The first and second authors are supported by the National Natural Science Foundation of China (Grant No. 10471050), by the National 973 Project of China (Grant No. 2006CB805902), by Guangdong Provincial Natural Science Foundation (Grant No. 7005795), and by University Special Research Foundation for Ph.D. Program (Grant No. 20060574002). The third and the fourth authors are supported by the National Natural Science Foundation of China Grant Nos. 10625105 and 10431060.


