Spreading speed and wavefronts for parabolic functional differential equations with spatio-temporal delays

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ABSTRACT

The existence of traveling wavefronts and the spreading speed for a kind of general reaction–diffusion system with spatio-temporal delays are investigated. We obtain the conclusion that there is \( c^* > 0 \) being the asymptotic speed of spread and the minimal speed for the system. Some more delicate analysis techniques have been developed to make the forms of the kernel function \( g(t,x) \) and the function \( f \) more flexible in our paper. A pair of super-solution and sub-solution have been constructed in order that the existence of traveling wavefronts can be guaranteed. There is a biological model as an example in this article, which is coincident with this system.

1. Introduction

In various dynamics, two key elements to the development process seem to be the appearance of traveling waves and the spreading speed (or, asymptotic speed of spread). Since the pioneer work of Fisher [1] and Kolmogorov et al. [2], the investigations on traveling wave solutions and asymptotic speeds of spread for various evolution systems modeling physical and biological phenomena are developed fast. Traveling waves were studied on nonlinear reaction–diffusion equations [3–5, 29, 30], integral and integro-differential systems [6–9], reaction–diffusion equations with time delays [10–13], lattice differential systems [14, 15] etc. The concept of asymptotic speed of spread was firstly introduced by Aronson [16] and Weinberger [17, 18], and has been studied more and more during these 30 years (see [7, 19, 10, 8, 14]).

In [11], Schaaf studied the asymptotic behavior and traveling wave solutions for a kind of parabolic functional differential equation with finite delay:

\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(u(t, x), u(t - \tau, x)), \quad \tau \in \mathbb{R}^+ := [0, \infty),
\]

where \( f \) satisfies \( f(0, 0) = f(1, 1) = 0 \). Schaff showed that (1.1) has a minimal speed \( c^* \), which is the spreading speed of the system as well. Recently, many works focus on the evolution systems with nonlocal effects which describe generally the joint effect of spatial diffusion and time delay on the dynamical properties of some biological or population systems, for example, see [20–23, 28]. One of the articles on this aspect is [21], where Wang, Li and Ruan [21] studied the following system which is a generalized form of (1.1):

\[
\frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + f(u(t, x), (g * u)(t, x)), \quad t > 0, x \in \mathbb{R},
\]

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where $u = (u_1, u_2, \ldots, u_n)^T$, $d = \text{diag}(d_1, d_2, \ldots, d_n)$ with $d_i > 0$ ($i = 1, 2, \ldots, n$), $f \in C(\mathbb{R}^{2n}, \mathbb{R}^n)$, and

$$(g * u)(t, x) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} g(t - s, x - y)u(s, y)dyds$$

$$= \int_{0}^{+\infty} \int_{-\infty}^{+\infty} g(s, y)u(t - s, x - y)dyds. \quad (1.3)$$

In [21], new iterative techniques are established for a class of integral operators when the reaction term satisfies the so-called $\gamma$-monotonicity, or $\gamma^*$-monotonicity, or $\gamma^{**}$-monotonicity. The existence of traveling wavefronts is established for the wave speed parameter $c \in I$, where $I$ is some sub-interval of $\mathbb{R}$, provided the existence of upper and lower solutions as $f$, $g$, $d$ satisfy some properties (see [21, Theorem 4.8]).

The purpose of this paper is to investigate the existence of the traveling wavefronts and spreading speed of system (1.2) as $n = 1$. We obtain conclusions that there is a $c^* > 0$, which can be calculated by solving a pair of functional equations, being the asymptotic speed of spread and the minimal speed for (1.2) under some assumptions which we shall give. Our methods here are motivated partly from [19,23,24]. We mention here that our assumption $(G2)$ is weaker than $(H2)$ in [23], thus the kernel function $g(t, x)$ in (1.2) could have more flexible form than that in [23]. Another challenging issue is that we hope that $f$ in (1.2) could have a general form, for example, $f$ satisfies (F1)–(F3) in the next paragraph, and thus we have to find other techniques to achieve this aim. In Sections 3 and 4, we first consider the problem with $f$ being smooth enough, and then by using the squeezing method, we obtain the existence of spreading speed and minimal wave speed $c^*$ successfully for $f$ satisfying (F1)–(F3). A pair of upper and lower solutions for the case $c > c^*$ are constructed concretely and delicately which guarantee the existence of traveling wavefronts in Section 4. As for $c = c^*$, we use a limit argument, and there are difficulties in proving that $\{U_k(z)\}$ is equi-continuous.

Assume that the function $f$ and the kernel $g$ satisfy the following hypotheses in the present article.

(F1) $f \in C(\mathbb{R}^2, \mathbb{R})$, there is unique $u^+ > 0$ such that $f(0, 0) = f(u^+, 0) = 0$ and $f(u, u) > 0$ for $u \in (0, u^+)$;

(F2) $f(r, s)$ is increasing on $s$ for $0 \leq r, s \leq u^+$;

(F3) for $(r_i, s_i) \in [0, R] \times [0, R]$ ($i = 1, 2$), there is $y_k$ such that

$$|f(r_1, s_1) - f(r_2, s_2)| \leq y_k(|r_1 - r_2| + |s_1 - s_2|),$$

for convenience we rewrite $y_k$ as $y$;

(G1) $g(t, x) \geq 0$, $g(t, -x) = g(t, x)$, $g$ is integrable with $\int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)dyds = 1$;

(G2) for any $c \geq 0$, there is $\tilde{\delta}(c) > 0$ such that $\int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)e^{-\lambda(\tilde{\delta}(c)y)}dyds < \infty$ for $\lambda \in (0, \tilde{\delta}(c))$ and

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)e^{-\tilde{\delta}(c)s}dyds = \infty.$$  

Remark 1.1. By the assumptions, model (1.2) has two nonnegative steady states: $u \equiv 0$, $u \equiv u^+$. It is easy to see that $\tilde{\delta}(c)$ in (G2) is increasing on $c$ and maybe $+\infty$.

2. Existence and comparison theorem of solutions

In this section we discuss the existence, uniqueness of solutions for the initial problem of (1.2). Furthermore, we establish a comparison result for the solutions of (1.2) with values in between the two steady states $u \equiv 0$ and $u \equiv u^+$.

Let $X = BUC(\mathbb{R}, \mathbb{R})$ be the space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the usual supremum norm $\| \cdot \|$, $X_+ = \{ \varphi \in X; \varphi(x) \geq 0, x \in \mathbb{R}\}$. Then $X_+$ is a closed cone of $X$ and $X$ is a Banach Lattice under the partial ordering $\leq_X$ induced by $X_+$.

Suppose $h : (0, \infty) \to [1, \infty)$ is a function satisfying the following conditions:

(H1) $h$ is continuous, nonincreasing and $h(0) = 1$;

(H2) $\frac{h(s + \theta)}{h(s)} \to 1$ uniformly for $s \in (-\infty, 0]$ as $\theta \to 0^+$;

(H3) $h(s) \to \infty$ as $s \to -\infty$.

Define

$$UC_h = \left\{ \phi : (-\infty, 0] \to X \text{ is continuous}, \sup_{s \leq 0} \frac{\| \phi(s) \|_X}{h(s)} < \infty \right\},$$

$$UC^+_h = \{ \phi \in UC_h; \phi(\theta) \geq 0 \text{ for } \theta \leq 0 \}.$$  

Similarly, $UC^+_h$ induces a partial ordering $\preceq_{UC^+_h}$ on $UC^+_h$. Let $UC_h$ be equipped with the norm $|\phi|_h = |\phi|_{UC_h} = \sup_{s \leq 0} \frac{|\phi(s)|_X}{h(s)}$ for $\phi \in UC_h$. According to Ruan and Wu [25], $(UC_h, \cdot |_h)$ is a Banach space. As usual we identify an element $\phi \in UC_h$ as a function from $(-\infty, 0] \times \mathbb{R} \to \mathbb{R}$ by $\phi(s, x) = \phi(x)(s)$. For any continuous function $u : (-\infty, b) \to X$, where $b > 0$, we define $u_t$ by $u_t(s) = u(t + s)$, $s \in (-\infty, 0]$. 

We define some subsets of \( \mathbb{X} \) and \( U_{C_h} \) by
\[
[0, u^+]_{\mathbb{X}} := \left\{ \varphi \in \mathbb{X}; \ 0 \leq \varphi(x) \leq u^+, \ x \in \mathbb{R} \right\},
\]
\[
[0, u^+]_{U_{C_h}} := \left\{ \phi \in U_{C_h}; \ 0 \leq \phi(t) \leq u^+ \text{ for } \theta \leq 0 \right\}.
\]
The solution of the following initial value problem of parabolic equation
\[
\frac{\partial w(t, x)}{\partial t} = d \frac{\partial^2 w(t, x)}{\partial x^2} - \gamma w(t, x)
\]
\[
w(0, x) = \varphi(x) \in \mathbb{X},
\]
is
\[
w(t, x) = [T(t)\varphi](x) := \frac{e^{-\gamma t}}{\sqrt{4\pi d t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4dt}} \varphi(y)dy.
\]
where \( T(t) : \mathbb{X} \to \mathbb{X} \) is an analytic semigroup with \( T(t) \mathbb{X}_+ \subset \mathbb{X}_+ \) for all \( t \geq 0 \), \( \gamma \) is given in (F3).

Rewrite (1.2) as
\[
\frac{\partial u(t,x)}{\partial t} = \frac{d \partial^2 u(t,x)}{\partial x^2} + \gamma u(t, x) = \gamma u(t, x) + f(u(t, x), (g * u)(t, x)).
\]
Define a functional \( F : U_{C_h} \to \mathbb{X} \) as follows:
\[
F(\phi)(x) = f(\phi(0, x), (g * \phi)(0, x)) + \gamma \phi(0, x)
\]
\[
= f \left( \phi(0, x), \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(s, x-y)\phi(-s, y)dyds \right) + \gamma \phi(0, x).
\]
Assume that \( \phi \in [0, u^+]_{U_{C_h}} \) is a given initial function, then the equivalent abstract integral form of (1.2) is
\[
u(t) = T(t)\phi(0, \cdot) + \int_{0}^{t} T(t-\tau)f(u_{\tau})d\tau, \quad t > 0,
\]
\[
u(t) = \phi(t, \cdot), \quad t \in (-\infty, 0].
\]

**Definition 2.1.** A continuous function \( v : (-\infty, b) \to \mathbb{X} \) is called a super-solution (sub-solution) of (1.2) on \([0, b)\) if
\[
v(t) \geq (\leq) T(t)\phi(0, \cdot) + \int_{0}^{t} T(t-\tau)f(v_{\tau})d\tau \quad \text{for } 0 \leq t < b.
\]

If \( v \) is both a super-solution and sub-solution on \([0, b)\), then it is said to be a (mild) solution of (1.2).

**Remark 2.1.** Assume that there is a bounded and continuous function \( v : \mathbb{R} \times (-\infty, b) \to \mathbb{X} \), with \( b > 0 \) and such that \( v \) is \( C^2 \) in \( x \in \mathbb{R}, C^1 \) in \( t \in (0, b) \), and
\[
\frac{\partial v(t, x)}{\partial t} \geq (\leq) d \frac{\partial^2 v(t, x)}{\partial x^2} + f(v(t, x), (g * v)(t, x))
\]
for \( (t, x) \in (0, b) \times \mathbb{X} \). Then by the fact that \( T(t) \mathbb{X}_+ \subset \mathbb{X}_+ \), it follows that (2.2) holds and hence \( v(t, x) \) is a super-solution (sub-solution) of (1.2) on \([0, b)\).

The main result of this section is the following.

**Theorem 2.1.** Suppose that the hypotheses (F1)–(F3) and (G1)–(G2) hold, then the following conclusions are valid:

(i) For any \( \phi \in [0, u^+]_{U_{C_h}} \), (1.2) has a unique solution \( u(t, x) = u(t, x; \phi) \) defined on \([0, \infty)\) such that \( u(t) \in [0, u^+]_{\mathbb{X}} \), \( u_t \in [0, u^+]_{U_{C_h}} \) for \( t \geq 0 \).

(ii) For any pair of super-solution \( \tilde{v}(t, x) \) and sub-solution \( v(t, x) \) of (1.2) on \( \mathbb{R} \times \mathbb{X} \) with \( 0 \leq \tilde{v}(t, x), v(t, x) \leq u^+ \) for \( (t, x) \in \mathbb{R} \times \mathbb{X} \) and \( 0 \leq \tilde{v}(s, x), v(s, x) \leq u^+ \) for \( (s, x) \in (-\infty, 0) \times \mathbb{X} \), there holds \( 0 \leq v(t, x) \leq \tilde{v}(t, x) \leq u^+ \) for \( (t, x) \in [0, +\infty) \times \mathbb{X} \).

**Proof.** Let \( T(t, \tau) = T(t-\tau), S(t, \tau) = T(t-\tau) \). Then by the hypotheses, one can verify that the conditions (T1)–(T2), (S1)–(S2) in Ruan and Wu [25] are satisfied.

By calculation and (F1), (F3), we obtain that
\[
|F(\phi_1) - F(\phi_2)|_{\mathbb{X}} = \sup_{x \in \mathbb{R}} |F(\phi_1)(x) - F(\phi_2)(x)| \leq 3\gamma |\phi_1 - \phi_2|_{U_{C_h}}
\]
as long as \( |\phi_1|_{U_{C_h}} \leq u^+, |\phi_2|_{U_{C_h}} \leq u^+ \). In view of **Definition 2.1** and **Remark 2.1**, \( u \equiv u^+ \) and \( u \equiv 0 \) is a pair of super-solution and sub-solution of (1.2). Let \( v^+ = u^+, v^- = 0 \), then (C3)–(C4) in Ruan and Wu [26] are satisfied. Furthermore, if
\( \phi_1, \phi_2 \in [0, u^+]_{U_C}, \) and \( \phi_1 \geq_{U_C} \phi_2, \) then we have from (F2) and (F3) that
\[
F(\phi_1)(x) - F(\phi_2)(x) = f(\phi_1(0, x), (g \ast \phi_1)(0, x)) - f(\phi_2(0, x), (g \ast \phi_2)(0, x)) + \gamma (\phi_1(0, x) - \phi_2(0, x))
\]
\[
= [f(\phi_1(0, x), (g \ast \phi_1)(0, x)) - f(\phi_2(0, x), (g \ast \phi_2)(0, x)) + \gamma (\phi_1(0, x) - \phi_2(0, x))]
\]
\[
+ [f(\phi_2(0, x), (g \ast \phi_2)(0, x)) - f(\phi_2(0, x), (g \ast \phi_2)(0, x))]
\geq 0.
\]
Thus \( F(\phi) \) is nondecreasing on \( \phi \in [0, u^+]_{U_C}. \) For any \( \phi_1, \phi_2 \in [0, u^+]_{U_C} \) with \( \phi_1 \geq_{U_C} \phi_2, \) we have the inequality
\[
[\phi_1(0) - \phi_2(0)] + l[F(\phi_1) - F(\phi_2)] \geq 0 \quad \text{for any } l \geq 0,
\]
which leads to
\[
\lim_{l \to 0^{+}} \text{dist} \{[\phi_1(0) - \phi_2(0)] + l[F(\phi_1) - F(\phi_2)], X_+\} = 0.
\]
For each \( b > 0, \) the existence and uniqueness of a solution \( u(t, x; \phi) \) on \( [0, b) \) follows from Theorem 5.2 in Ruan and Wu [25] with \( S(t, s) = T(t, s) = T(t - s) \) for \( t \geq s \geq 0 \) and \( u^+ = u^+, v^- = 0. \) Note \( 0 \leq u(t, x; \phi) \leq u^+ \) on \( [0, b) \) hence the maximal interval of existence is \( [0, b). \)

We now prove the conclusion (ii). Some \( \bar{u}, \bar{u} \in [0, u^+]_{U_C} \) and \( \bar{u} \leq_{U_C} \bar{u} \leq u^+, \) it follows from Theorem 5.2 in Ruan and Wu [25] that
\[
0 \leq u(t, x; \bar{u}) \leq u(t, x; \bar{u}) \leq u^+ \quad \text{for } x \in \mathbb{R}, \ t \geq 0.
\]
Again by applying Theorem 5.2 in Ruan and Wu [25] with
\[
u_-(t, x) = v(t, x), \quad u^+(t, x) \equiv u^+;
\]
\[
u_-(t, x) \equiv 0, \quad u^+(t, x) = \tilde{v}(t, x),
\]
respectively, we get
\[
u(t, x) \leq u(t, x; \bar{u}) \leq u^+ \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R},
\]
\[
0 \leq u(t, x; \bar{u}) \leq \tilde{v}(t, x) \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R},
\]
from which it follows that \( \bar{u}(t, x) \leq \tilde{v}(t, x) \) for all \( (t, x) \in [0, \infty) \times \mathbb{R}. \) \( \blacksquare \)

3. The spreading speed

The spreading speed (or, asymptotic speed of spread) is an important concept to describe the asymptotic properties of a given system at the time \( t = \infty. \) In fact, it is a threshold constant \( c^* > 0 \) which gives an important description of the long time behaviors of the investigated systems either for \( c \in (0, c^*) \) or for \( c \in (c^*, \infty). \) Taking (1.2) as an example, the spreading speed \( c^* \) is a number in the sense that:
\[
\forall c > c^*, \quad \lim_{t \to \infty, |x| \geq ct} u(t, x) = 0,
\]
\[
\forall 0 < c < c^*, \quad \lim_{t \to \infty, |x| \geq ct} u(t, x) = u^+,
\]
where
\[
\lim_{t \to \infty, |x| \geq ct} u(t, x) = \lim_{t \to \infty, |x| \geq ct} \sup u(t, x) = \lim_{t \to \infty, |x| \geq ct} \inf u(t, x),
\]
and \( \lim_{t \to \infty, |x| \leq ct} u(t, x) \) is defined similarly (see [9]). In this section, we shall show the existence of \( c^* \) and establish a pair of equations to calculate \( c^*. \)

We now consider (1.2) when \( f \) satisfies the assumptions (F4)–(F8).

(F4) \( f \in C^1(\mathbb{R}^2, \mathbb{R}), f(0, 0) = 0 \) for \( u^+ > 0, f(u, u) > 0 \) for \( u \in (0, u^+); \)

(F5) \( f^2(0, s) \geq 0 \) for \( s \in [0, u^+], f^2(0, 0) > 0 \) for \( f^2(0, 0) > 0, f^2(0, 0) > 0, \)

where \( f^2(0, 0) = \frac{\partial f(r, s)}{\partial t} |_{(r, s) = (0, 0)}, f^2(0, 0) = \frac{\partial f(r, s)}{\partial t} |_{(r, s) = (0, 0)}; \)

(F6) \( f(r, s) \leq f^2(0, 0)r + f^2(0, 0) \) for \( r \leq s \leq u^+; \)

(F7) \( f \) for any small \( \varepsilon > 0, \) there is \( \delta > 0 \) such that \( f(r, s) \geq f^2(0, 0)r + f^2(0, 0)(1 - \varepsilon) \) for \( r, s \in [0, \delta]; \)

(F8) there is a \( \eta \in (0, 1) \) such that for any \( \delta \in (1 - \eta, 1), \) there exists unique \( \alpha \in (0, u^+) \) such that \( f(\alpha, \alpha \delta) = 0. \)

Remark 3.1. We mention here that our assumptions (F4)–(F8) are reasonable, and there are many population models being coincident with them such as the models in Al-Omar & Gourley [20], Weng [22] and Zhao [23]. Assumption (F8) is proposed the first time, but there are many functions satisfying it. For example, let \( f(r, s) = -ar + b[1 - r]s, 0 < a < b, \) by simple calculation, for \( 0 < \eta < 1, \) \( \delta \in (1 - \eta, 1), \) we have \( \alpha = 1 - \frac{a}{bd} \in (0, u^+), \) where \( u^+ = \frac{b - a}{b} \), thus the function \( f(r, s) \) satisfies the assumption (F8).

Linearizing (1.2) at zero solution when \( f \) satisfies (F4)–(F8), we obtain
\[
\frac{\partial u}{\partial t}(t, x) = d \Delta u(t, x) + f^2(0, 0)u(t, x) + f^2(0, 0)(g \ast u)(t, x).
\]
Assume (G1) and (G2) hold, define a function $p$ with two parameters $\lambda$ and $c$:

$$p(\lambda, c) := d\lambda^2 - c\lambda + f'_1(0, 0) + f'_2(0, 0) \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)e^{-\lambda(cs+y)}dyds,$$

(3.1)

where $\lambda \in [0, \bar{\delta}(c))$ and $p$ is continuous and differentiable on $\lambda$ and $c$.

It is easy to obtain

$$\frac{\partial p}{\partial \lambda} = 2d\lambda - c + f'_2(0, 0) \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)[-c(s + y)]e^{-\lambda(cs+y)}dyds,$$

$$\frac{\partial^2 p}{\partial \lambda^2} = 2d + f'_2(0, 0) \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)(cs + y)^2e^{-\lambda(cs+y)}dyds > 0,$$

$$p(0, c) = f'_1(0, 0) + f'_2(0, 0) > 0,$$

$$p(\lambda, 0) = d\lambda^2 + f'_1(0, 0) + f'_2(0, 0) \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)e^{-\lambda y}dyds \geq 0,$$

$$p(\lambda, +\infty) = -\infty \text{ for any given } \lambda > 0,$$

$$\lim_{\lambda \to \bar{\delta}(c)} p(\lambda, c) = +\infty \text{ for any given } c > 0,$$

$$\frac{\partial p}{\partial c} = -\lambda + f'_2(0, 0) \int_0^{+\infty} \int_{-\infty}^{+\infty} (-\lambda s)g(s, y)e^{-\lambda(cs+y)}dyds < 0 \text{ for } \lambda > 0.$$  

In view of the above properties of $p$, we obtain Lemma 3.1.

**Lemma 3.1.** There exists a unique pair of $(\lambda^*, c^*)$ such that

(i) $p(\lambda^*, c^*) = 0$, $\frac{\partial p}{\partial \lambda}(\lambda^*, c^*) = 0$;

(ii) $p(\lambda, c) > 0$ holds for $0 < c < c^*$ and $\lambda \in [0, \bar{\delta}(c))$;

(iii) for $c > c^*$, there are two zeros $0 < \lambda_{a(c)} < \lambda_{b(c)} < \bar{\delta}(c)$ for $p(\lambda, c) = 0$. Furthermore, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ with $0 < \lambda_{a(c)} < \lambda_{a(c)} + \varepsilon < \lambda_{b(c)}$, we have

$$p(\lambda_{a(c)} + \varepsilon, c) < 0. \quad (3.2)$$

We wonder whether $c^*$ is the spreading speed for system (1.2) when $f$ satisfies (F4)–(F8). Motivated by Zhao and Xiao [23], a limit procedure will be used in what follows. Firstly consider the following relevant equation with finite delay $\tau > \tau_0 > 0$:

$$\frac{\partial u^{(\tau)}(t, x)}{\partial t} = d\Delta u^{(\tau)}(t, x) + f(u^{(\tau)}(t, x), \int_0^t \int_0^{+\infty} g(s, y)u^{(\tau)}(t - s, x - y)dyds), \quad (3.3)$$

where $\tau_0$ satisfies $\int_0^{\tau_0} \int_{-\infty}^{+\infty} g(s, y)dyds \geq 1 - \eta$ and

$$f'_1(0, 0) + f'_2(0, 0) \int_0^{\tau_0} \int_{-\infty}^{+\infty} g(t, x)dxdt > 0,$$

where $\eta$ is the number defined in (F8).

Obviously, $0$ is an equilibrium of Eq. (3.3). Another equilibrium $L_\tau$ of (3.3) is decided by using (F8) and the equation

$$f(L_\tau, L_\tau) \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)dyds = 0 \quad (3.4)$$

for $\tau \geq \tau_0$ and it is obviously that $L_\tau \to u^+$ as $\tau \to +\infty$.

Let $C := C([-\tau, 0], \mathbb{R})$. For any $x \in \mathbb{R}$, $u^{(\tau)}(\cdot, x) \in C$ is defined by

$$u^{(\tau)}(\theta, x) = u^{(\tau)}(t + \theta, x), \quad \forall \theta \in [-\tau, 0].$$

Assume that the initial function is $\phi(\theta, x)$, $\theta \in [-\tau, 0], x \in \mathbb{R}$. Then $u^{(\tau)}(t, x) = u^{(\tau)}(t, x; \phi)$ denotes the unique solution of (3.3) satisfying $u^{(\tau)}(\theta, x) = \phi(\theta, x)$ for $\theta \in [-\tau, 0], x \in \mathbb{R}$.

Define a functional $f^{(\tau)}(\phi)(x) = f(\phi(0, x), \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)\phi(-s, x - y)dyds)$. By (F4) and (F5), (F8), it is easy to verify that $f^{(\tau)}$ satisfies the two assumptions (F1) and (F2) in [19, Theorem 5.1], then there is a $c^*_{\tau}$ which is the spreading speed of (3.3). Furthermore by (F6) and (F7), we can estimate $c^*_{\tau}$ by using the method proposed by [19, Section 5.1]. Consider the following equation

$$\frac{dV}{dt} = d\lambda^2V(t) + f'_1(0, 0)V(t) + f'_2(0, 0) \int_0^{t} \left( \int_{-\infty}^{+\infty} g(s, y)e^{\lambda y}dy \right) V(t - s)ds,$$  

(3.5)
where $\lambda$ is a parameter. The characteristic equation for the above differential equation is

$$\chi_t - d\lambda^2 - f'_1(0, 0) - f'_2(0, 0) \int_0^\tau \left( \int_{-\infty}^{+\infty} g(s, y)e^{s\lambda}dy \right) e^{-\lambda t}ds = 0.$$  \tag{3.6}

We know that (3.6) admits a real root $\chi_t = \chi_t(\lambda) \geq d\lambda^2 + f'_1(0, 0)$, which is the principal eigenvalue of (3.5). Furthermore, we have from (3.6) that

$$\chi_t(0) = f'_1(0, 0) + f'_2(0, 0) \int_0^\tau \int_{-\infty}^{+\infty} g(s, y)e^{-\lambda t}dyds > 0,$$

and thus (C7) in [19] is satisfied. Now we obtain that

$$c^*_\tau = \inf_{\lambda > 0} \Phi_t(\lambda) = \inf_{\lambda > 0} \chi_t(\lambda) / \lambda \geq \inf_{\lambda > 0} \left\{ d\lambda + \frac{f'_1(0, 0)}{\lambda} \right\} \text{ for } \tau > \tau_0,$$

where $\Phi_t(\lambda) := \chi_t(\lambda) / \lambda$ with $\Phi_t(\infty) = \infty$, $\lim_{\lambda \to 0} \Phi_t(\lambda) = \infty$. Let $c := \Phi_t(\lambda)$. By [19, Lemma 3.8] there is unique $\lambda^*_\tau$ such that $c^*_\tau = \Phi_t(\lambda^*_\tau)$. Define

$$p_t(\lambda, c) = d\lambda^2 - c\lambda + f'_1(0, 0) + f'_2(0, 0) \int_0^\tau \int_{-\infty}^{+\infty} g(s, y)e^{-\lambda(t+s)}dyds.$$

Noting that $p_t(\lambda, c) = 0$ leads to $\frac{dc}{d\lambda} = -\frac{dp_t}{dp}$, and comparing $p_t(\lambda, c) = 0$ with (3.6), we know that $(\lambda^*_\tau, c^*_\tau)$ can be determined by

$$p_t(\lambda^*_\tau, c^*_\tau) = 0. \quad \frac{dp_t}{d\lambda}(\lambda^*_\tau, c^*_\tau) = 0.$$  \tag{3.8}

Remark 3.2. By an analogous computation to $p(\lambda, c)$, $p_t(\lambda, c)$ has the following properties.

(i) $p_t(\lambda^*_\tau, c^*_\tau) = 0$, $\frac{dp_t}{d\lambda}(\lambda^*_\tau, c^*_\tau) = 0$;

(ii) $p_t(\lambda, c) > 0$ holds for $0 < c < c^*_\tau$ and $\lambda \in [0, \delta_t(c)]$, where $\delta_t(c) \geq \tilde{\delta}(c)$ is a number similar to $\tilde{\delta}(c)$ in (G2);

(iii) for $c > c^*_\tau$, there exist two zeros $\lambda_{a(t, c)}, \lambda_{b(t, c)}$ such that $0 < \lambda_{a(t, c)} < \lambda_{b(t, c)} < \delta_t(c)$ for $p_t(\lambda, c) = 0$. Furthermore, we have

$$p_t(\lambda, c) < 0 \text{ for any } \lambda \in (\lambda_{a(t, c)}, \lambda_{b(t, c)}),$$

$$p_t(\lambda, c) > 0 \text{ for any } \lambda \in [0, \lambda_{a(t, c)}) \cup (\lambda_{b(t, c)}, \delta_t(c)).$$  \tag{3.9}

By the above discussion and [19, Theorem 5.1], we have the following conclusions.

Proposition 3.1. Let $\tau > \tau_0$ and $L$ be decided by (3.4), and assume that (F4)–(F8) and (G1)–(G2) hold. Then for any $\phi^{(*)} : 0 \leq \phi^{(*)}(\theta, x) < L$, with $\theta \in [-\tau, 0], x \in \mathbb{R}$, the following statements are valid.

(i) For any $c > c^*_\tau$, if $\phi^{(*)} : 0 \leq \phi^{(*)}(\cdot, x) < L_t$ and $\phi^{(*)}(\cdot, x) = 0$ for $x$ outside a bounded interval, then

$$\lim_{t \to +\infty} L_t \kappa^{(*)}(t, x; \phi^{(*)}) = 0.$$  \tag{3.11}

(ii) For any $0 < c < c^*_\tau$, if $\phi^{(*)} : 0 \leq \phi^{(*)}(\cdot, x) \leq L_t$ with $\phi^{(*)}(\theta, x) \neq 0$ for $\theta \in [-\tau, 0]$, then

$$\lim_{t \to +\infty} L_t \kappa^{(*)}(t, x; \phi^{(*)}) = L_t.$$  \tag{3.12}

Remark 3.3. For Eq. (3.3), we can have analogous definitions and conclusions to Definition 2.1, Remark 2.1 and Theorem 2.1, but we omit the details here.

We are interested in the relation of $(\lambda^*_\tau, c^*_\tau)$ to $(\lambda^*, c^*)$, and thus we have the following results.

Proposition 3.2. \lim_{t \to +\infty} (\lambda^*_\tau, c^*_\tau) = (\lambda^*, c^*)

Proof. From the above discussion, we have known that $c^*_\tau$ is decided by (3.8). Obviously $c^*_\tau$ depends on $\tau$. We have from the second equation in (3.8) that

$$c^*_\tau = 2d\lambda^*_\tau - f'_2(0, 0) \int_0^\tau \int_{-\infty}^{+\infty} g(s, y)(c^*_\tau s + y)e^{-\lambda^*_\tau(t+s)}dyds,$$

which leads to a conclusion that $c^*_\tau$ is increasing for $\tau > 0$. Thus there are only two cases for $c^*_\tau$.

(a) $\lim_{\tau \to +\infty} c^*_\tau = +\infty$.

(b) There is $\tilde{c} < +\infty$ such that $\lim_{\tau \to +\infty} c^*_\tau = +\infty$.

We first claim that (a) is not true. Otherwise there is $\tau > 0$ such that $c^*_\tau > c^*$, then $p_t(\lambda^*, c^*) < p(\lambda^*, c^*) = 0$, which contradicts (ii) of Remark 3.2. Thus only (b) holds. We claim that $(\lambda^*_\tau)_{\tau \geq \tau_0}$ is bounded. Otherwise, there is $\{\tau_j\}$ such that
Let \( \lambda^*_t \to +\infty \) as \( t \to +\infty \). Noting that \( c^*_t \) is bounded for \( t \), for \( \lambda^*_t \) is sufficiently large, we obtain
\[
\frac{d(\lambda^*_t)^2}{\lambda^*_t} - c^*_t' + f'_1(0, 0) + f'_2(0, 0) \int_0^t \int_{\mathbb{R}} g(s, y)e^{-y(c^*_t+y)} \, dy \, ds > 0,
\]
which contradicts \( p_t(\lambda^*_t, c^*_t) = 0 \). Thus \( \lambda^*_t \) is bounded. We have from (3.8) that \( \lambda^*_t = \lambda(c^*_t) \). Noting that \( c^*_t \) is increasing for \( t > 0 \) and \( \lambda(c^*_t) \) is continuous, we know
\[
\lim_{t \to +\infty} (\lambda^*_t, c^*_t) = (\tilde{\lambda}, \tilde{c}).
\]
To claim the proposition, we want to show that
\[
p(\tilde{\lambda}, \tilde{c}) = 0.
\]
In [23], they believe that it is valid naturally, but by (G2), we have to prove it by delicate analysis. Since \( \tilde{c} \geq c^*_t \), then \( p_t(\lambda^*_t, \tilde{c}) \leq p_t(\lambda^*_t, c^*_t) = 0 \) and there are \( \lambda_{a(t, \tilde{c})} \) and \( \lambda_{b(t, \tilde{c})} \) such that (3.9) holds, thus
\[
p_t(\lambda_{a(t, \tilde{c})}, \tilde{c}) = p_t(\lambda_{b(t, \tilde{c})}, \tilde{c}) = 0, \quad \lambda^*_t \in [\lambda_{a(t, \tilde{c})}, \lambda_{b(t, \tilde{c})}].
\]
The fact that \( p_t(\lambda, c) \) is increasing on \( \tau \) implies the conclusion that \( \lambda_{a(t, \tilde{c})} \) is increasing on \( \tau \) and \( \lambda_{b(t, \tilde{c})} \) is decreasing on \( \tau \), thus there are \( \hat{\lambda}_{a(t, \tilde{c})} \) and \( \hat{\lambda}_{b(t, \tilde{c})} \) such that \( \hat{\lambda}_{a(t, \tilde{c})} = \lim_{\tau \to +\infty} \lambda_{a(t, \tilde{c})} \) and \( \hat{\lambda}_{b(t, \tilde{c})} = \lim_{\tau \to +\infty} \lambda_{b(t, \tilde{c})} \). By (3.10), we obtain
\[
\hat{\lambda} \in [\hat{\lambda}_{a(t, \tilde{c})}, \hat{\lambda}_{b(t, \tilde{c})}] \subseteq [\lambda_{a(t, \tilde{c})}, \lambda_{b(t, \tilde{c})}].
\]
By (3.9) and (3.13), we have
\[
p_t(\hat{\lambda}, \tilde{c}) = d(\hat{\lambda})^2 - c\hat{\lambda} + f'_1(0, 0) + f'_2(0, 0) \int_0^\tau \int_{\mathbb{R}} g(s, y)e^{-\hat{\lambda}(c^*_t+y)} \, dy \, ds \leq 0
\]
for any \( \tau > 0 \), then there is \( \tilde{M} \) which is independent on \( \tau \) such that
\[
\int_{\mathbb{R}} g(s, y)e^{-\tilde{\hat{\lambda}}(c^*_t+y)} \, dy < \tilde{M} \quad \text{for any } \tau > \tau_0.
\]
Therefore
\[
\int_{\mathbb{R}} g(s, y)e^{-\tilde{\hat{\lambda}}(c^*_t+y)} \, dy < \tilde{M},
\]
which implies that \( p(\tilde{\lambda}, \tilde{c}) \) is meaningful and \( p(\tilde{\lambda}, \tilde{c}) \leq 0 \). We claim that \( p(\tilde{\lambda}, \tilde{c}) > 0 \) is impossible. Suppose \( p(\tilde{\lambda}, \tilde{c}) = -\tilde{A} < 0 \), noting that the function \( p \) is continuous on \( (\lambda, c) \), then we have \( \varepsilon > 0 \) such that if \( (\lambda, c) \) satisfies \( |\lambda - \tilde{\lambda}| + |c - \tilde{c}| < \varepsilon \), there is
\[
p(\lambda, c) < -\frac{\tilde{A}}{2} < 0.
\]
By (3.10), there is \( \tau > \tau_0 \) sufficiently large that \( |\lambda^*_t - \tilde{\lambda}| + |c^*_t - \tilde{c}| < \varepsilon \) such that
\[
p(\lambda^*_t, c^*_t) < -\frac{\tilde{A}}{2} < 0.
\]
Together with \( p_t(\lambda^*_t, c^*_t) \leq p(\lambda^*_t, c^*_t) \), we have \( p_t(\lambda^*_t, c^*_t) < 0 \), which is a contradiction with \( p_t(\lambda^*_t, c^*_t) = 0 \). Then (3.11) is valid, and the proposition is proved.

The following lemma and theorem show that \( c^* \) is the spreading speed of (1.2).

Lemma 3.2. Let \( \Gamma(t - r, x - y) = \frac{e^{-\gamma(t-r)}}{\sqrt{4\pi d(t-r)}} e^{-\frac{(x-y)^2}{4d(t-r)}}, \) then
\[
\int_{\mathbb{R}} \Gamma(t - r, x - y)e^{-\hat{\lambda}y} \, dy = e^{-\hat{\lambda}x} e^{(d\hat{\lambda}^2 - \gamma)(t-r)}.
\]
The proof is easy, we omit it here.

Theorem 3.1. Suppose the hypotheses (F4)–(F8) and (G1)–(G2) hold, then the following statements are valid.
(i) If \( \phi \in [0, u^*]_{[\mathbb{R}]} \) with \( \phi(\theta, x) = 0 \) for \( (\theta, x) \) outside a bounded set, then
\[
\lim_{t \to +\infty} u(t, x; \phi) = 0 \quad \text{for any } c > c^*.
\]
(ii) If \( \phi \in [0, u^+]_{\text{loc}} \) with \( \phi(\theta, \cdot) \not\equiv 0 \) for \( \theta \in (-\infty, 0] \), then

\[
\lim_{t \to -\infty, |x| \leq \varepsilon} u(t, x; \phi) = u^+ \quad \text{for any } 0 < c < c^*.
\]

**Proof.** (1) Let \( c > c^* \) be fixed and \( \bar{c} \in (c^*, c), \bar{\lambda} > 0 \) be chosen such that

\[
p(c, \bar{\lambda}) < 0.
\]  

(3.14)

Let \( \bar{u}(t, x) := \min \{ \beta e^\xi (\bar{t} - \text{sgn}(x)), u^+ \} \). It is obvious that \( \bar{u}(t, x) \) is not \( C^2 \) in \( x \in \mathbb{R} \) and \( C^1 \) in \( t \in \mathbb{R}^+ \). We shall prove that \( \bar{u}(t, x) \) is the super-solution of (1.2) when \( f \) satisfies (F4)--(F8) by using the abstract form

\[
\bar{u}(t, x) \geq T(t)\phi(0, \cdot) + \int_0^t T(t - \tau)F(\bar{u}_\tau)\,d\tau,
\]

where \( T(t) \) is defined by (2.1), and \( \gamma \) in \( T(t) \) is defined by \( \gamma = \max_{0 \leq \tau < u^+} \{ |f'_1(\tau, s)|, |f'_2(\tau, s)| \} \).

Since \( \phi \) has a compact support, we can choose \( \beta \) large sufficiently such that

\[
\phi(t, x) \leq \min \{ \beta e^\xi (\bar{t} - \text{sgn}(x)), u^+ \} = \bar{u}(t, x).
\]

If \( \bar{c}t - |x| \geq \frac{1}{\lambda} \ln \left( \frac{|x|^2}{\bar{\lambda}} \right) \), then \( \bar{u}(t, x) = u^+ \), obviously

\[
T(t)\phi(0, \cdot) + \int_0^t T(t - \tau)F(\bar{u}_\tau)\,d\tau \leq u^+.
\]  

(3.15)

If \( \bar{c}t - |x| < \frac{1}{\lambda} \ln \left( \frac{|x|^2}{\bar{\lambda}} \right) \), then \( \bar{u}(t, x) = \beta e^{\bar{c}\xi (\bar{t} - \text{sgn}(x))} \), we claim that

\[
T(t)\phi(0, \cdot) + \int_0^t T(t - \tau)F(\bar{u}_\tau)\,d\tau \leq \bar{u}(t, x) = \beta e^{\bar{c}\xi (\bar{t} - \text{sgn}(x))}.
\]  

(3.16)

In fact, assume that \( x \geq 0 \), by **Lemma 3.2**, we have

\[
T(t)\phi(0, \cdot) \leq \int_{-\infty}^{+\infty} \Gamma(t, x - y) \beta e^{-\bar{c}y} \,dy = \beta e^{-\bar{c}x} e^{(d\bar{c} - \gamma)t}.
\]  

(3.17)

Let

\[
A := \gamma + f'_1(0, 0) + f'_2(0, 0) \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)e^{-\bar{c}(s + y)}\,ds\,dy.
\]

Then by (F6) and **Lemma 3.2**, we have

\[
\int_0^t T(t - \tau)F(\bar{u}_\tau)\,d\tau \leq A\beta \int_0^t \int_{-\infty}^{+\infty} \Gamma(t - \tau, x - y) e^{\bar{c}(\bar{t} - \bar{\lambda} - \bar{y})} \,dy\,d\tau
\]

\[
= \frac{A\beta e^{-\bar{c}x}}{\bar{\lambda}c + \gamma - d\bar{\lambda}^2} e^{(d\bar{c} - \gamma)t} \left[ e^{(\bar{\lambda}c + \gamma - d\bar{\lambda}^2)t} - 1 \right].
\]  

(3.18)

By (3.17) and (3.18), we obtain

\[
T(t)\phi(0, \cdot) + \int_0^t T(t - \tau)F(\bar{u}_\tau)\,d\tau \leq \beta e^{-\bar{c}x} e^{(d\bar{c} - \gamma)t} \left[ 1 - \frac{A}{\bar{\lambda}c + \gamma - d\bar{\lambda}^2} \right] + \frac{A\beta}{\bar{\lambda}c + \gamma - d\bar{\lambda}^2} e^{-\bar{c}x + \bar{\lambda}c t}.
\]

By (3.14), there are \( \bar{\lambda}c + \gamma - d\bar{\lambda}^2 - A = -p(\bar{\lambda}, \bar{c}) > 0 \) and \( d\bar{\lambda}^2 - \gamma < \bar{c}\bar{\lambda} \), thus

\[
T(t)\phi(0, \cdot) + \int_0^t T(t - \tau)F(\bar{u}_\tau)\,d\tau \leq \beta e^{-\bar{c}x} e^{\bar{\lambda}c t} \left[ -p(\bar{\lambda}, \bar{c}) + A \right] \frac{\bar{\lambda}c + \gamma - d\bar{\lambda}^2}{\bar{\lambda}c + \gamma - d\bar{\lambda}^2}.
\]

(3.19)

If \( x < 0 \), one can discuss by a similar way. Therefore, (3.16) holds, (3.15) and (3.16) lead to a conclusion that \( \bar{u}(t, x) \) is a super-solution of (1.2) when \( f \) satisfies (F4)--(F8). Thus for any \( c > c^* \), we obtain

\[
\lim_{t \to -\infty, |x| \leq \varepsilon} u(t, x) \leq \lim_{t \to -\infty, |x| \leq \varepsilon} \bar{u}(t, x) = 0.
\]

(2) Let \( c \in (0, c^*) \) be fixed. By **Proposition 3.2**, there exists a sufficiently large \( \tau^* \) such that \( c^* > c \) for \( \tau > \tau^* \). Let

\[
\phi^{(c)}(\theta, x) := \min \{ \phi(\theta, x), L_\tau \}, \quad \forall (\theta, x) \in [\tau, 0] \times \mathbb{R}.
\]
Since $u(t, x; \phi)$ is a super-solution of equation (3.3) on the interval $[-\tau, 0]$, $u^{(t)}(t, x; \phi^{(t)})$ is the solution of (3.3) with $u_0^{(t)} = \phi^{(t)}$, and thus can be regarded as a sub-solution of (3.3). By a comparison theorem for (3.3) similar to Theorem 2.1 (see Remark 3.3), we have

$$u(t, x; \phi) \geq u^{(t)}(t, x; \phi^{(t)}), \quad \forall (t, x) \in [0, +\infty) \times \mathbb{R}.$$  

Since $\phi(\theta, \cdot) \neq 0$ for $\theta \in (-\infty, 0]$, we have $\phi^{(t)}(\theta, \cdot) \neq 0$ for $\theta \in [-\tau, 0]$. By Proposition 3.1, it follows that

$$u^+ \geq \lim_{t \to +\infty, |x| \leq ct} u(t, x; \phi) \geq \lim_{t \to +\infty, |x| \leq ct} u^{(t)}(t, x; \phi^{(t)}) = L_t, \quad \forall \tau \geq t'.$$

Letting $\tau \to +\infty$, we then obtain $\lim_{t \to +\infty, |x| \leq ct} u(t, x; \phi) = u^+$ since $\lim_{t \to +\infty} L_t = u^+$. Thus the proof is complete. \qed

In the above discussion, we need assumptions (F4)–(F5) in order to use Theorem 5.1 in [19] for obtaining the existence of spreading speed $c^*$ for (1.2) and (F6)–(F7) in order to estimate $c^*$. It is natural to ask whether a general form of $f$ which satisfies the weaker assumptions (F1)–(F3) could lead to analogous conclusions. The following auxiliary condition (F9) will offer us this possibility.

(F9) For $u^+$ satisfying $f(u^+, u^+) = 0$, there exist $\bar{f}, \tilde{f}$ such that $\bar{f}(r, s) \leq f(r, s) \leq \tilde{f}(r, s)$ for $0 \leq r, s \leq u^+$, and (F4)–(F8) hold for $\bar{f}, \tilde{f}$ with $\bar{f}(0, 0) = \tilde{f}(0, 0) = \bar{f}_1(0, 0) = \tilde{f}_1(0, 0)$, $\bar{f}_2(0, 0) = \tilde{f}_2(0, 0)$. But the condition that $f(r, s) \in C^1$ with $r, s \in [0, u^+]$ is not necessary here.

We will handle (1.2) for the case which $f$ satisfies (F1)–(F3) and (F9) by using comparison method. Considering the equations

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \tilde{f}(v(t, x), g \ast v(t, x))$$

and

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \bar{f}(v(t, x), g \ast v(t, x)),$$

we try to prove that $c^*$ is the spreading speed of (1.2) provided that it is the speed of (3.19) and (3.20) simultaneously.

**Theorem 3.2.** Suppose (F1)–(F3) and (F9), (G1)–(G2) hold. Then there is $c^* > 0$ such that the following statements hold.

(i) If $\phi \in [0, u^+]_{C^0}$ with $\phi(\theta, x) = 0$ for $(\theta, x)$ outside a bounded set, then

$$\lim_{t \to +\infty, |x| \geq ct} u(t, x; \phi) = 0 \quad \text{for any } c > c^*.$$  

(ii) If $\phi \in [0, u^+]_{C^0}$ with $\phi(\theta, \cdot) \neq 0$ for $\theta \in (-\infty, 0]$, then

$$\lim_{t \to +\infty, |x| \leq ct} u(t, x; \phi) = u^+ \quad \text{for any } 0 < c < c^*.$$  

Moreover, $c^*$ is determined by the system

$$p(\lambda, c) = 0, \quad \frac{\partial p}{\partial \lambda}(\lambda, c) = 0,$$

where

$$p(\lambda, c) = d\lambda^2 - c\lambda + \tilde{f}_1(0, 0) + \tilde{f}_2(0, 0) \int_0^\infty \int_{-\infty}^{+\infty} g(s, y) e^{-\lambda(c+y)} dy ds$$

$$= d\lambda^2 - c\lambda + \bar{f}_1(0, 0) + \bar{f}_2(0, 0) \int_0^\infty \int_{-\infty}^{+\infty} g(s, y) e^{-\lambda(c+y)} dy ds.$$  

**Proof.** Let $\hat{v}(t, x) = \hat{u}(t, x; \phi)$ be the solution of the following equation

$$\frac{\partial v}{\partial t} = d \frac{\partial^2 v}{\partial x^2} + \tilde{f}(v(t, x), g \ast v(t, x)).$$

By (F9) and Theorem 2.1, we have $0 \leq \hat{u}(t, x) \leq u^+$ and

$$\frac{\partial \hat{v}}{\partial t} \geq d \frac{\partial^2 \hat{v}}{\partial x^2} + f(\hat{u}(t, x), g \ast \hat{u}(t, x)).$$

Thus $\hat{v}(t, x)$ is a super-solution of (1.2). Note that $u(t, x; \phi)$ is a sub-solution of (1.2), and we have from Theorem 2.1 that

$$0 \leq u(t, x; \phi) \leq \hat{u}(t, x; \phi) \leq u^+.$$
where \( u(t, x; \phi) \) is the solution of (1.2) when \( f \) satisfies (F1)–(F3) and (F9). By (i) of Theorem 3.1, (i) of Theorem 3.2 is valid. By the similar discussion and using \( f \), (ii) of Theorem 3.1 implies (ii) of Theorem 3.2. The proof is complete.

4. Traveling wavefronts

In this section, assume that (F1)–(F3) and (G1)–(G2) hold. Let us consider the traveling wave solution of (1.2) with the form \( u(t, x) = U(x + ct), \) \( c > 0, \) connecting the two equilibria \( u \equiv 0, u \equiv u^+ \). Let \( z = x + ct \), then \( U(z) \) satisfies

\[
cU(z) = dU''(z) + f(U(z), (g * U)(z)), \quad z \in \mathbb{R},
\]

where

\[
(g * U)(z) = \int_{0}^{+\infty} \int_{-\infty}^{+\infty} g(s, y)U(z - y - cs)dyds.
\]

Here, \( U(z) \) also satisfies the boundary condition

\[
U(-\infty) = 0, \quad U(+\infty) = u^+.
\]

Here, we pay our attention to the nondecreasing solutions of (4.1) and (4.3).

Let \( C_{[0,u^+]}(\mathbb{R}, \mathbb{R}) := \{ \varphi : \varphi \in C(\mathbb{R}, \mathbb{R}), 0 \leq \varphi(z) \leq u^+ \text{ for } z \in \mathbb{R} \} \),

\( Y := \{ \varphi : \varphi \in C_{[0,u^+]}(\mathbb{R}, \mathbb{R}), \varphi' \text{ and } \varphi'' \in L^\infty(\mathbb{R}) \} \).

Define

\[
[QU](z) := \gamma U(z) + f(U(z), (g * U)(z)),
\]

then (4.1) can be written as

\[
-dU'' + cU'(z) + \gamma U(z) = [QU](z),
\]

where \( \gamma > 0 \) is defined in (F3).

Note that (4.5) is equivalent to the following integral equation

\[
U(z) = \int_{-\infty}^{+\infty} K(z, s, c)[QU](s)ds,
\]

where

\[
K(z, s, c) = \begin{cases} \frac{1}{\xi} e^{\lambda_1(s - c)} & s \leq z \\ \frac{1}{\xi} e^{\lambda_2(s - c)} & s \geq z \end{cases}
\]

and

\[
\xi(c) := d(\lambda_2(c) - \lambda_1(c)), \quad \lambda_1(c) = \frac{c - \sqrt{c^2 + 4\gamma d}}{2d}, \quad \lambda_2(c) = \frac{c + \sqrt{c^2 + 4\gamma d}}{2d},
\]

here \( \lambda_1(c) \) and \( \lambda_2(c) \) are increasing on \( c \).

Assume that \( f \in C(\mathbb{R}^2, \mathbb{R}) \), thus Theorem 5.1 in [23,24] cannot be applied here to obtain the existence of traveling wavefronts. From [21], we know that we can reach the aim by constructing a pair of upper solution and lower solution.

**Definition 4.1.** A continuous function \( \rho : \mathbb{R} \to \mathbb{R} \) is called an upper solution of (4.1) if \( \rho' \) and \( \rho'' \) exist almost everywhere, and they are essentially bounded on \( \mathbb{R} \) and satisfy

\[
-d\rho''(z) + c\rho'(z) \geq f(\rho(z), (g * \rho)(z)), \quad \text{a.e. on } \mathbb{R}.
\]

A lower solution of (4.1) is defined in a similar way by reversing the inequality in (4.8).

From (F3), we can easily verify that

\[
f(\varphi_1(z), (g * \varphi_2(z))) + \gamma \varphi_2(z) \geq f(\varphi_1(z), (g * \varphi_1(z))) + \gamma \varphi_1(z),
\]

with \( \varphi_1, \varphi_2 \in C_{[0,u^+]}(\mathbb{R}, \mathbb{R}), \varphi_2 \geq \varphi_1 \). Note that the conditions (H0), (H1), (H2), (H3) in [21] are satisfied by conditions (F1)–(F3) and (G1)–(G2), thus by [21, Theorem 4.8] and [27, Theorem 3.6’], we have the following theorem.

**Theorem 4.1.** If (4.1) has an upper solution \( \bar{\rho} \in C_{[0,u^+]}(\mathbb{R}, \mathbb{R}) \cap Y \) which is increasing and a lower solution \( \rho \in Y \) such that for \( z \in \mathbb{R} \)

(C1) \( 0 \leq \lim_{z \to -\infty} \bar{\rho}(z) < u^+, \lim_{z \to -\infty} \bar{\rho}(z) = u^+, \bar{\rho}'(z^+) \leq \bar{\rho}'(z^-);\)
(C2) \( \rho \neq 0, \lim_{z \to -\infty} \rho(z) = 0 \) and \( \rho'(z^+) \geq \rho'(z^-) \);

(C3) \( \bar{\rho}(z) \leq \rho(z) \).

Then there exists at least one monotone solution of (4.1) satisfying (4.3), i.e., (1.2) has a traveling wavefront connecting 0 and \( u^+ \).

Remark 4.1. Wu and Zou [27] pointed out that their original requirements in [13] for the super-solution and the sub-solution cannot guarantee the monotonicity of the iteration and they have to make the assumptions stronger in [27]. Thus we have to add the assumptions \( \bar{\rho}'(z^+) \leq \rho'(z^-), \bar{\rho}'(z^+) \geq \rho'(z^-) \) besides the assumptions in [21, Theorem 4.8].

From Theorem 4.1, the existence of the traveling wavefront of (4.1) and (4.3) will be guaranteed provided that we find an upper solution and a lower solution of (4.1) satisfying Theorem 4.1. To this end, we give assumption (F10), which is motivated from [11].

(F10) There exist \( \bar{f}, \hat{f} \) such that (F9) is satisfied, moreover \( \bar{f}(r, s) \in C^{1,v}(\mathbb{R}^2, \mathbb{R}) \) for some Hölder index \( v \in (0, 1) \), where 
\[
C^{1,v}(\mathbb{R}^2, \mathbb{R}) := \{ \psi : \psi \in C^1(\mathbb{R}^2, \mathbb{R}) \text{ and there is } k > 0 \text{ such that } |D\psi(p_1) - D\psi(p_2)| \leq k|p_1 - p_2|^v, \forall p_1 = (r_1, s_1) \in \mathbb{R}^2 \text{ and } l = l_1 + l_2 = 1, l_1, l_2 = 0 \text{ or } 1).\]

Let \( p(\lambda, c) \) is given by (3.21), \( \lambda_{ad(c)} \) be defined in Lemma 3.1 and \( \lambda_0 := \lambda_{ad(c)} \), thus \( p(\lambda_0, c) = 0 \). Assume that \( v \) is defined in (F10), and let \( \varepsilon \leq \frac{1}{2} \lambda_0 \varepsilon \) be small such that \( p(\lambda_0, \varepsilon) \) is a small solution of \( (4.1) \). Define
\[
\bar{U}(z) = \min \left\{ u^+, u^+ e^{\lambda_0 z} \right\}, \quad \underline{U}(z) = \max \left\{ 0, \sigma_1 e^{\lambda_0 z} - \sigma_2 e^{\lambda_0 z} \right\}, \quad \forall z \in \mathbb{R},
\]
where \( \sigma_1 \) and \( \sigma_2 \) are chosen suitably so that \( 0 < \sigma_1 \leq \sigma_2, \sigma_2 > 1, \sup_{z \in \mathbb{R}} U(s) \leq \bar{U}(z), z \in \mathbb{R} \). Furthermore, we have \( \hat{U}'(0^+) \leq \bar{U}'(0^-), \underline{U}'(z_0^+) \geq \underline{U}'(z_0^-) \), where \( z_0 := \frac{1}{2} \ln \left( \frac{2\lambda_0}{\sigma_2} \right) \leq 0 \) is the root of \( \sigma_1 e^{\lambda_0 z} - \sigma_2 e^{\lambda_0 z} = 0 \). Thus \( \bar{U}(z) \) and \( \underline{U}(z) \) satisfy the conditions in Theorem 4.1.

Lemma 4.1. Assume that (F1)–(F3), (F10) and (G1)–(G2) hold, for \( c > c^*, \bar{U}(z) \) and \( \underline{U}(z) \) are a pair of upper solution and lower solution of (4.1).

Proof. If \( z \geq 0 \), then \( \bar{U}(z) = u^+ \), thus by (F2)
\[
d\bar{U}''(z) - c\bar{U}'(z) + f(\bar{U}(z), (g * \bar{U})(z)) = f\left( u^+, \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)\bar{U}(z - y - cs)dyds \right) \leq f(u^+, u^+) = 0.
\]

If \( z < 0 \), \( \bar{U}(z) \leq u^+ e^{\lambda_0 z}, z \in \mathbb{R} \), by (F5) and (F10) we have
\[
d\bar{U}''(z) - c\bar{U}'(z) + f(\bar{U}(z), (g * \bar{U})(z)) \leq d\bar{U}''(z) - c\bar{U}'(z) + \hat{f}(\bar{U}(z), (g * \bar{U})(z))
\]
\[= \left[ d(\lambda_0)^2 e^{\lambda_0 z} - c\lambda_0 e^{\lambda_0 z} u^+ + \hat{f}(u^+ e^{\lambda_0 z}, (g * \bar{U}(z))) \right]
\]
\[\leq \left[ d(\lambda_0)^2 e^{\lambda_0 z} - c\lambda_0 e^{\lambda_0 z} + \hat{f}_1(0, 0) e^{\lambda_0 z} u^+ + \hat{f}_2(0, 0) (g * \bar{U})(z) \right]
\]
\[\leq u^+ \left[ d(\lambda_0)^2 - c\lambda_0 + \hat{f}_1(0, 0) + \hat{f}_2(0, 0) \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y) e^{-\lambda_0(y + cs)}dyds \right] e^{\lambda_0 z} = 0.
\]
Thus \( \bar{U}(z) \) is an upper solution of (4.1).

Next, we claim that \( \underline{U}(z) \) is a lower solution of (4.1). Note that
\[
\underline{U}(z) \geq \sigma_1 e^{\lambda_0 z} - \sigma_2 e^{\lambda_0 z} =: k_1(z) \quad \text{for } z \in \mathbb{R},
\]
and
\[
k_1(z) = \begin{cases} 
> 0, & z < z_0 \\
= 0, & z = z_0 \\
< 0, & z > z_0.
\end{cases}
\]

If \( z \geq 0 \), then \( \underline{U}(z) = 0 \), thus by (F2)
\[
d\underline{U}''(z) - c\underline{U}'(z) + f(\underline{U}(z), (g * \underline{U})(z)) \geq f(0, 0) = 0.
\]

If \( z < 0 \), then \( \underline{U}(z) = \sigma_1 e^{\lambda_0 z} - \sigma_2 e^{\lambda_0 z} \), thus by (F2), (F10), (F5) and (4.9) that
\[
d\underline{U}''(z) - c\underline{U}'(z) + f(\underline{U}(z), (g * \underline{U})(z)) \geq d\underline{U}''(z) - c\underline{U}'(z) + \hat{f}(\underline{U}(z), (g * \underline{U})(z))
\]
\[\geq d(\lambda_0)^2 \sigma_1 e^{\lambda_0 z} - d(\lambda_0)^2 \sigma_2 e^{\lambda_0 z} - c\lambda_0 \sigma_1 e^{\lambda_0 z} + c\lambda_0 \sigma_2 e^{\lambda_0 z}
\]
\[\geq \sigma_1 e^{\lambda_0 z} - \sigma_2 e^{\lambda_0 z} \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y) \left[ \sigma_1 e^{\lambda_0(z - y - cs)} - \sigma_2 e^{\lambda_0(z - y - cs)} \right] dyds.
\]
In the following, we estimate the value of the term
\[
\tilde{f} \left( \sigma_1 e^{\lambda z} - \sigma_2 e^{\lambda z}, \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y) \left[ \sigma_1 e^{\lambda (z-y-cs)} - \sigma_2 e^{\lambda (z-y-cs)} \right] dy ds \right).
\]

By (G2), there are \( M_1, M_2 \) such that
\[
\int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y) \sigma_1 e^{\lambda (z-y-cs)} dy ds = M_1 \sigma_1 e^{\lambda z},
\]
\[
\int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y) \sigma_2 e^{\lambda (z-y-cs)} dy ds = M_2 \sigma_2 e^{\lambda z},
\]
then
\[
\tilde{f} \left( \sigma_1 e^{\lambda z} - \sigma_2 e^{\lambda z}, \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y) \left[ \sigma_1 e^{\lambda (z-y-cs)} - \sigma_2 e^{\lambda (z-y-cs)} \right] dy ds \right) = \tilde{f} \left( k_1(z), k_2(z) \right),
\]
where \( k_2(z) := M_1 \sigma_1 e^{\lambda z} - M_2 \sigma_2 e^{\lambda z}. \) Noting \( \sigma_1 e^{\lambda z} > \sigma_2 e^{\lambda z} \) for \( z < z_0 \) and \( \sigma_2 > \sigma_1 > 0, \) we have
\[
\sqrt{k_1^2(z) + k_2^2(z)} = \sqrt{(\sigma_1 e^{\lambda z} - \sigma_2 e^{\lambda z})^2 + (M_1 \sigma_1 e^{\lambda z} - M_2 \sigma_2 e^{\lambda z})^2}
\]
\[
\leq \sqrt{(\sigma_1 e^{\lambda z})^2 + (M_1 \sigma_1 e^{\lambda z} + M_2 \sigma_2 e^{\lambda z})^2}
\]
\[
\leq \sqrt{(\sigma_1 e^{\lambda z} + M_1 \sigma_1 e^{\lambda z} + M_2 \sigma_2 e^{\lambda z})^2}
\]
\[
\leq (1 + M_1 + M_2) \sigma_2 e^{\lambda z}.
\]

Then by the mean value theorem, there is \( \theta(z) \in (0, 1) \) such that
\[
\tilde{f}(k_1(z), k_2(z)) = \tilde{f}_1(\theta(z) k_1(z), \theta(z)k_2(z)) k_1(z) + \tilde{f}_2(\theta(z) k_1(z), \theta(z)k_2(z)) k_2(z)
\]
\[
= \left[ \tilde{f}_1(\theta(z) k_1(z), \theta(z)k_2(z)) - \tilde{f}_1(0, 0) k_1(z) \right]
\]
\[
+ \left[ \tilde{f}_2(\theta(z) k_1(z), \theta(z)k_2(z)) - \tilde{f}_2(0, 0) k_2(z) \right]
\]
\[
+ \tilde{f}_1(0, 0) k_1(z) + \tilde{f}_2(0, 0) k_2(z).
\]

Without loss of generality, we may choose \( \sigma_2 > 1, \) and therefore \( \sigma_2^\nu < \sigma_2. \) From (F10), we know that \( \tilde{f} \in C^{1,\nu}(\mathbb{R}^2, \mathbb{R}), \) and thus from [14.13], we have
\[
\left| \tilde{f}(k_1(z), k_2(z)) - \tilde{f}_1(0, 0) k_1(z) - \tilde{f}_2(0, 0) k_2(z) \right|
\]
\[
\leq \left| \tilde{f}_1(\theta(z) k_1(z), \theta(z)k_2(z)) - \tilde{f}_1(0, 0) k_1(z) \right| + \left| \tilde{f}_2(\theta(z) k_1(z), \theta(z)k_2(z)) - \tilde{f}_2(0, 0) k_2(z) \right|
\]
\[
\leq \tilde{k} \left( \sqrt{k_1^2(z) + k_2^2(z)} \right)^\nu \left| k_1(z) + k_2(z) \right|
\]
\[
\leq \tilde{k}(1 + M_1 + M_2) \nu \sigma_2^\nu e^{\lambda z^2} \left[ \sigma_1 e^{\lambda z} + M_1 \sigma_1 e^{\lambda z} + M_2 \sigma_2 e^{\lambda z} \right]
\]
\[
\leq \tilde{k}(1 + M_1 + M_2) \nu \sigma_2^\nu e^{\lambda z^2} \left[ 1 + M_1 \sigma_1 e^{\lambda z} + M_2 \sigma_2 e^{\lambda z} \right].
\]

Note that \( \sigma_2 e^{\lambda z} < \sigma_1 \) if \( z < z_0 \leq 0 \) and \( \epsilon \leq \frac{1}{2} \lambda a \nu, \) we have
\[
\sigma_1 \sigma_2 e^{\lambda z^2} \frac{e^{\lambda z}}{e^{\lambda z}} \leq \sigma_1 \frac{e^{\lambda z^2} e^{\lambda z}}{e^{\lambda z}} = \sigma_1 \frac{e^{\lambda z^2} e^{\lambda z}}{e^{\lambda z}} < \sigma_1^2 e^{\lambda z^2} \leq \sigma_2^2 e^{\lambda z^2} < \sigma_1^2 e^{\lambda z^2}.
\]

Therefore, by (4.15) and (4.16), there is \( M = \tilde{k} \left[ \sigma_1^2 (1 + M_1 + M_2)^{1+\nu} \right] \) such that
\[
\left| \tilde{f}(k_1(z), k_2(z)) - \tilde{f}_1(0, 0) k_1(z) - \tilde{f}_2(0, 0) k_2(z) \right| \leq \tilde{k}(1 + M_1 + M_2)^{1+\nu} \sigma_1^2 e^{\lambda z^2} = M e^{\lambda z^2}.
\]

Note that \( p(\lambda, c) = 0, \) then we have from (4.11), (4.12), (4.14) and (4.17) that
\[
dU''(z) - cU''(z) + f(U(z), (g * U)(z)) \geq \frac{-e^{\lambda z^2}}{\sigma_2 p(\lambda, c) + M}.
\]

Since \( M \) is independent of \( \sigma_2 \) and \( p(\lambda, c) < 0, \) then we can choose \( \sigma_2 \) large enough such that \( -e^{\lambda z^2} [\sigma_2 p(\lambda, c) + M] \geq 0, \) thus \( U(z) \) is a lower solution of (4.1). The proof is complete. \qed

**Theorem 4.2.** Let \( c^* \) be defined in **Theorem 3.2**, and suppose the hypotheses (F1)–(F3) and (F10), (G1)–(G2) hold, then the following two statements are valid:

(i) for any \( c \geq c^*, \) (4.1) has a monotone traveling wave solution \( U(x + ct) \) connecting 0 and \( u^+; \)

(ii) for any \( c \in (0, c^*), \) (4.1) admits no monotone traveling wave solution \( U(x+ct) \) connecting 0 to \( u^+ \) with \( U(\cdot) \in C_{[0, +\infty)}(\mathbb{R}, \mathbb{R}). \)
**Proof.** For \( c > c^* \), from Theorem 4.1 and Lemma 4.1, we can easily prove (i).

For \( c = c^* \), we use a limit argument. Comparing with [23], since \( f \) is generalized, we have to overcome more difficulties in proving the fact \( \{ U_k(z) \}_{k=1}^\infty \) is equi-continuous. Let \( \{ c_k \} \subset (c^*, c^* + 1) \) with \( \lim_{k \to \infty} c_k = c^* \). Since \( c_k > c^* \), (4.1) with \( c = c_k \) admits a nondecreasing solution \( U_k(z) \) such that \( U_k(-\infty) = 0 \), \( U_k(+\infty) = u^+ \). Without loss of generality, we may assume that \( U_k(0) = \frac{1}{2} u^+ \). We have from (4.6)

\[
U_k(z) = \int_{-\infty}^{+\infty} K(z, s, c_k)[QU_k](s) ds,
\]

(4.18)

Obviously \( \{ U_k \} \) is uniformly bounded and we claim that \( \{ U_k \} \) is equi-continuous on \( \mathbb{R} \).

For simplicity, we write \( \lambda_1(c_k), \lambda_2(c_k) \) as \( \lambda_{1k}, \lambda_{2k} \) respectively. Since \( c^* + 1 > c_k > c^* \), by (7.7) there are

\[
\lambda_1(c^*) < \lambda_{1k} < \lambda_1(c^* + 1) < 0 < \lambda_2(c^*) < \lambda_{2k} < \lambda_2(c^* + 1).
\]

(4.19)

Moreover,

\[
0 < \zeta_* := d[\lambda_2(c^*) - \lambda_1(c^* + 1)] < \zeta(c_k) = d(\lambda_{2k} - \lambda_{1k}).
\]

(4.20)

For any \( z, z' \in \mathbb{R} \), we assume that \( z > z' \). Note that there is a unique \( z_{1k} \in (z', z) \) which is concerned in \( c_k \) such that

\[
e^{\lambda_{1k}(z - z_{1k})} = e^{\lambda_{2k}(z' - z_{1k})},
\]

that is

\[
e^{\lambda_{1k}(z-z')} < e^{\lambda_{2k}(z' - s)} \quad \text{for} \quad s < z_{1k}, \quad e^{\lambda_{1k}(z-s)} > e^{\lambda_{2k}(z' - s)} \quad \text{for} \quad s > z_{1k}.
\]

Note that there is \( \tilde{M} > 0 \) such that \( |QU_k(z)| \leq \tilde{M} \) for any \( z \in \mathbb{R}, \ k \in \mathbb{N} \), we obtain

\[
\left| \int_{-\infty}^{+\infty} \left[ K(z, s, c_k) - K(z', s, c_k) \right] QU_k(s) ds \right| \leq \tilde{M} \int_{-\infty}^{+\infty} \left[ K(z, s, c_k) - K(z', s, c_k) \right] ds
\]

\[
\leq \frac{\tilde{M}}{\zeta(c_k)} \left[ \int_{-\infty}^{z'} \left( e^{\lambda_{1k}(z'-s)} - e^{\lambda_{1k}(z-s)} \right) ds + \int_{z'}^{z_{1k}} \left( e^{\lambda_{2k}(z'-s)} - e^{\lambda_{1k}(z-s)} \right) ds \right]
\]

\[+ \frac{\tilde{M}}{\zeta(c_k)} \left[ \int_{z_{1k}}^{z'} \left( e^{\lambda_{1k}(z-s)} - e^{\lambda_{2k}(z'-s)} \right) ds + \int_{z}^{+\infty} \left( e^{\lambda_{2k}(z-s)} - e^{\lambda_{2k}(z'-s)} \right) ds \right]
\]

\[= \frac{\tilde{M}}{\zeta(c_k) \lambda_{1k}} \left( e^{\lambda_{1k}(z-z_{1k})} - 1 \right) + \frac{2\tilde{M}}{\zeta(c_k) \lambda_{2k}} \left( 1 - e^{\lambda_{2k}(z-z_{1k})} \right).
\]

By (4.19) and (4.20),

\[
\left| \int_{-\infty}^{+\infty} \left[ K(z, s, c_k) - K(z', s, c_k) \right] QU_k(s) ds \right| \leq \frac{\tilde{M} \lambda_1(c^* + 1)}{\zeta \lambda_1(c^*)} \left( e^{\lambda_1(c^*)(z-z_{1k})} - 1 \right) + \frac{2\tilde{M}}{\zeta \lambda_2(c^*)} \left( 1 - e^{\lambda_2(c^*)(z-z_{1k})} \right)
\]

\[\leq \frac{2\tilde{M}}{\zeta \lambda_1(c^*)} \left( e^{\lambda_1(c^*)(z-z_{1k})} - 1 \right) + \frac{2\tilde{M}}{\zeta \lambda_2(c^*)} \left( 1 - e^{\lambda_2(c^*)(z-z_{1k})} \right),
\]

here \( \zeta_*, \lambda_1(c^*), \lambda_1(c^* + 1), \tilde{M}, \lambda_2(c^*), \lambda_2(c^* + 1) \) are independent on \( c_k \), then for any \( \varepsilon > 0 \), we can choose \( \delta(\varepsilon) \) such that if \( |z' - z| < \delta \), then \( |U_k(z) - U_k(z')| < \varepsilon \) for any \( k \in \mathbb{N} \), thus \( \{ U_k \} \) is equi-continuous.

Using Arzela–Ascoli theorem and the standard diagonal method, we can obtain a subsequence of functions \( \{ U_{k_m}(z) \} \) which converges to \( U^*(z) \) as \( m \to \infty \), uniformly for \( z \) in any bounded subset of \( \mathbb{R} \). Clearly, \( U^*(z) \) is nondecreasing and \( U^*(0) = \frac{1}{2} u^+ \). By the dominated convergence theorem and (4.18), it follows that

\[
U^*(z) = \int_{-\infty}^{+\infty} K(z, s, c^*)[QU^*](s) ds.
\]
The left is to prove that \( U^*(z) \) satisfies (4.3). Since \( U^*(z) \) is nondecreasing, then there are \( A_1, A_2 \) with \( 0 \leq A_1 \leq A_2 \leq u^+ \) such that

\[
\lim_{z \to -\infty} U^*(z) = A_1, \quad \lim_{z \to +\infty} U^*(z) = A_2.
\]

By (4.6), there is

\[
\lim_{z \to -\infty} U^*(z) = \lim_{z \to +\infty} \frac{1}{\zeta(c^*)} \left[ \int_z^{+\infty} e^{\lambda_2(c^*) (z-t)} [QU^*](s)ds + \int_{-\infty}^{-z} e^{\lambda_1(c^*) (t-z)} [QU^*](s)ds \right]
\]

In view of L’Hospital rule,

\[
A_1 = [QA_1] \frac{1}{\zeta(c^*)} \left( \frac{1}{\lambda_2(c^*)} - \frac{1}{\lambda_1(c^*)} \right) = [QA_1] \frac{-1}{d\lambda_1(c^*) \lambda_2(c^*)},
\]

since \( -\frac{1}{\lambda_1(c^*) \lambda_2(c^*)} = -\frac{d}{\lambda_1(c^*)}, \) \( QA_1 = yA_1 + f(A_1, A_1), \) thus \( f(A_1, A_1) = 0. \) Using the same way, we obtain that \( f(A_2, A_2) = 0. \) Noting that \( U \equiv 0 \) and \( U \equiv u^+ \) are the only two constant solutions of \( f(u, u) = 0, \) it is easy to see that \( A_1 = 0 \) and \( A_2 = u^+. \) Therefore, \( U(x + ct) \) is a monotone traveling wavefront of (1.2) connecting 0 to \( u^+. \)

For the case \( c < c^*, \) we use reduction to absurdity. Suppose there is \( c_1 < c^* \) such that \( U(x + c_1 t) \) is the monotone traveling wavefronts connecting 0 to \( u^+. \) Since \( U(+\infty) = u^+, \) then there exists \( h \) such that \( U(x) > \frac{1}{2}u^+ \) for \( x \geq h. \) Let \( \phi(x, t) = U(x + c_1 t), \) then \( \phi \in [0, u^+] \cup \mathbb{R}, \phi(\cdot, t) \not\equiv 0 \) for \( t \in (-\infty, 0), \) and \( u(t, x, \phi) = U(x + c_1 t). \) By (ii) of Theorem 3.2, we have

\[
\lim_{t \to -\infty, |x| \leq ct} u(t, x, \phi) = \lim_{t \to -\infty, |x| \leq ct} U(x + c_1 t) = u^+ \text{ for any } c \in (0, c^*).
\]

Let \( c = -c_2 t, 0 < c_1 < c_2 < c^*, \) then

\[
\lim_{t \to -\infty, |x| \leq -c_2 t} U(x + c_1 t) = \lim_{t \to +\infty} U((c_1 - c_2) t) = U(-\infty) = u^+,
\]

which is a contradiction. The proof is complete. \( \square \)

5. Application

In this section, we shall introduce an example to illustrate the application of our main results.

Example 1. Ruan and Xiao [26] have recently presented a host–vector model for a disease without immunity in which the current density of infectious vectors is related to the number of infectious hosts at earlier time:

\[
\frac{\partial u}{\partial t}(t, x) = d\Delta u(t, x) - au(t, x) + b[1 - u(t, x)] \int_{-\infty}^{t} \int_{-\infty}^{+\infty} g(t, s, x, y) u(s, y)dyds.
\]

Zhao and Xiao [23] consider the special case of the above model:

\[
\frac{\partial u}{\partial t}(t, x) = d\Delta u(t, x) - au(t, x) + b[1 - u(t, x)] \int_{-\infty}^{t} \int_{-\infty}^{+\infty} g(t - s, x - y) u(s, y)dyds.
\] (5.1)

They [23] assume that

(H1) \( b > a > 0, g(s, y) = g(s, -y) \) and \( \int_{0}^{+\infty} \int_{-\infty}^{+\infty} g(s, y)dyds = 1; \)

(H2) \( \int_{0}^{+\infty} \int_{-\infty}^{+\infty} g(s, y)e^{-\lambda(y+c\phi)}dyds < +\infty \) for all \( c \geq 0 \) and \( \lambda \geq 0. \)

The two equilibria of (5.1) are \( u \equiv 0, u \equiv 1 - \frac{c}{b}. \) We mention here that (H2) is more strict. For example, let \( g(s, y) = \frac{1}{\tau}e^{-\frac{s}{\sqrt{4\pi}}e^{-\frac{y^2}{\tau}}} \), then

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(s, y)e^{-\lambda(y+c\phi)}dyds = \int_{0}^{+\infty} \frac{1}{\tau}e^{(\lambda^2-\lambda\phi)^{1/2}} dyds,
\]

thus (H2) is not satisfied. But if we let \( \delta(c) = \frac{c^{1/2}+\lambda^{1/2}}{2}, \) then (G2) is satisfied.

For (5.1), \( f(r, s) = -ar + b(1 - r)s \) satisfies the assumptions (F4)–(F8) in this article, let \( g(t, x) \) satisfy the assumptions (G1)–(G2), by Theorems 3.1 and 4.2, we have a conclusion that there is \( c^* > 0 \) being the asymptotic speed of spread and the
minimal speed for model (5.1). Furthermore, let
\[ p(\lambda, c) = d\lambda^2 - c\lambda - a + b \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y) e^{-\lambda(y+c)} dy ds, \]
c* is decided by \( p(\lambda^*, c^*) = 0, \frac{dp}{d\lambda}(\lambda^*, c^*) = 0. \)
Furthermore, one can consider a more general form of (5.1), that is
\[ \frac{\partial u}{\partial t}(t, x) = D\Delta u(t, x) - H_1(u(t, x)) + b(1 - u(t, x)) H_2 \left( \int_0^{+\infty} \int_{-\infty}^{+\infty} g(t-s, x-y) u(s, y) dy ds \right), \quad (5.2) \]
where \( H_1, H_2 \in C(C(\mathbb{R}^2, C^+)), H \in C^1(v(\mathbb{R}^2, C^+)) \) such that for \( 0 \leq r, s \leq 1 - \frac{a}{b}, -ar + b(1-r)s - \tilde{H}(r, s) \leq b(1-r)H_2(s) - H_1(r) \leq -ar + b(1-r)s + \tilde{H}(r, s). \)
Assume \( H_1, H_2, \tilde{H} \) satisfy the following assumptions,
(H3) \( H_2(s) \) is increasing on \( s \) for \( 0 \leq s \leq 1 - \frac{a}{b}; \)
(H4) for \( r_1, r_2 \in [0, R] \), there is \( \gamma_R \) such that
\[ |H_i(r_1) - H_i(r_2)| \leq \gamma_R |(r_1 - r_2)|, \quad (i = 1, 2); \]
(H5) \( \tilde{H}(0, 0) = \tilde{H}(0, 0) = \tilde{H}(1 - \frac{a}{b}, 1 - \frac{a}{b}) = \tilde{H}(1 - \frac{a}{b}, 1 - \frac{a}{b}) = 0, \tilde{H}_1(0, 0) = \tilde{H}_1(0, 0) = \tilde{H}_2(0, 0) = \tilde{H}_2(0, 0) = 0; \)
(H6) there is \( \eta > 0 \) such that for any \( \delta \in (1 - \eta, 1) \), a pair of \( \alpha, \beta \in (0, 1 - \frac{a}{b}) \) exist such that \( \tilde{H}(\alpha, \beta) = a\alpha - b(1 - \alpha)\beta \), \( \tilde{H}(\beta, \beta) = a\beta + b(1 - \beta)\beta \), maybe \( \alpha \neq \beta; \)
(H7) for \( 0 \leq r, s \leq 1 - \frac{a}{b}, \tilde{H}(r, s) \leq brs \) and \( \tilde{H}(r, r) \leq -ar + b(1-r)r \) hold.
Then \( f(r, s) = b(1-r)H_2(s) - H_1(r), \tilde{f}(r, s) = -ar + b(1-r)s - \tilde{H}(r, s) \) and \( \tilde{f}(r, s) = -ar + b(1-r)s + \tilde{H}(r, s), \) by (H3), (H4) and (H5), \( f(r, s) \) satisfies (F1)–(F3) in our paper, \( f \) and \( \tilde{f} \) satisfy (F4)–(F10) provided that \( \tilde{H}(r, s), H(r, s) \) satisfy (H5)–(H7). By using our Theorems 3.2 and 4.2, \( c^* \) is the asymptotic speed as well as the minimal speed of (5.2).

References