Existence of traveling wave solutions in nonlinear delayed cellular neural networks

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Abstract

This paper is concerned with the existence of traveling wave solutions of cellular neural network systems distributed in the one-dimensional lattice $\mathbb{Z}$. The dynamics of each given cell depends on itself and its nearest right neighbor cell where delays exist in self-feedback and neighborhood interaction. Under appropriate assumptions, we can prove the existence of traveling wave solutions whose output function is not piecewise linear.

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1. Introduction

The cellular neural network system was first proposed by Chua and Yang [3,4] as an achievable alternative to fully-connected neural networks in electric circuit systems, so it is also called CY-CNN system. In recent years, CNN systems have been extensively studied and found to have many important applications to a broad scope of problems arising from various fields such as image and video signal processing, robotic and biological visions etc, we refer the readers to [2–6] for examples on some practical applications. The cellular neural network is formed by basic circuit units called cells. In the implementation of networks, time delays are inevitably encountered because of the finite switching speed of signal transmission. So cellular neural networks with delay were introduced, see, for example [5–7,10]. One of the models distributed in the one-dimensional integer lattice $\mathbb{Z}$ without input, is proposed in [10] by

$$\frac{dx_i(t)}{dt} = -x_i(t) + z + \alpha \int_0^\tau K_1(u)f(x_i(t-u)) + \beta \int_0^\tau K_2(u)f(x_{i+1}(t-u))du$$

for $i \in \mathbb{Z}$. It is assumed that the dynamics of each given cell depends on itself and its nearest right neighbor cell where delay exists in self-feedback and neighborhood interaction. The coefficients $\alpha > 0, \beta > 0$ of the signal output function $f$ constitute the so-called space-invariant template that measures the synaptic weights of self-feedback and neighborhood interaction. $\tau > 0$ is a constant, $K_1 : [0, \tau] \to [0, +\infty)$ are piecewise-continuous functions satisfying

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\[
\int_0^T K_i(u)du = 1, \quad i = 1, 2. \tag{1.2}
\]

The quantity \( z \) is called a threshold or bias term and is related to the independent voltage sources in electric circuits. A typical output function \( f \) in (1.1) which is a piecewise-linear function, see, for examples [3–10], is defined by

\[
f(x) = \begin{cases} 
1 & \text{if } x \geq 1, \\
x & \text{if } |x| \leq 1, \\
-1 & \text{if } x \leq -1. 
\end{cases} \tag{1.3}
\]

Our focus here is a traveling wave solution given by

\[
x_i(t) = \phi(i - ct) \quad i \in \mathbb{Z} \text{ and } t \in \mathbb{R} \tag{1.4}
\]

for a wave profile \( \phi(s), s = i - ct \in \mathbb{R} \), with a given wave speed \( c \in \mathbb{R} \). To investigate such traveling waves is important in a number of applications including image processing, see [1,7–10]. Clearly such a profile with \( z = 0 \) must satisfy the functional differential equation

\[
-c\phi'(s) = -\phi(s) + \alpha \int_0^T K_1(u)f(\phi(s + cu))du + \beta \int_0^T K_2(u)f(\phi(s + 1 + cu))du. \tag{1.5}
\]

Assuming that the synaptic connection is sufficiently large so that

\[
\alpha + \beta > 1, \tag{1.6}
\]

then there are three equilibria of (1.5):

\[
x^- = -(\alpha + \beta), \quad x^0 = 0, \quad x^+ = \alpha + \beta. \tag{1.7}
\]

Recently, Weng and Wu [10] investigated the existence of traveling wave solutions for (1.1) with (1.3), which satisfy different asymptotic boundary conditions, for example,

\[
\lim_{s \to -\infty} \phi(s) = x^0, \quad \lim_{s \to \infty} \phi(s) = x^+. \tag{1.8}
\]

We note that the typical output function \( f \) in [3–10] is linear on \([-1, 1]\). Naturally, a question arises: does there exist traveling wave solutions in (1.1) if \( f \) is nonlinear on \([-1, 1]\)? To the best of our knowledge, it seems that little has been done for this case. This paper is to answer this question and consider the existence of traveling wave solutions of (1.1) under the following hypotheses:

(H\(_f\)) \( f \) is a continuous odd function on \((-\infty, +\infty)\) and satisfies:

1. \( f(x) = 1 \) for \( x \geq 1 \);
2. \( f(0) = 0 \) and \( f \) is differentiable at \( x = 0 \), \( \mu = f'(0) \geq 1 \);
3. \( f \) is nondecreasing and \( f(x) \leq f'(0)x \) on \([0, 1]\).

Under the above hypotheses (1.6) and (H\(_f\)), we see that \( x^- \), \( x^0 \) and \( x^+ \) are still equilibria of (1.5). Moreover, we have

\[
f'(0)x - f(x) = o(x) \quad \text{as } x \to 0. \tag{1.9}
\]

In fact, noting that \( f(0) = 0 \), one has

\[
\lim_{x \to 0} \frac{f'(0)x - f(x)}{x} = f'(0) - \lim_{x \to 0} \frac{f(x) - f(0)}{x} = 0,
\]

hence \( f'(0)x - f(x) = o(x) \).

2. Preliminaries

To investigate the existence of monotone traveling waves of (1.1), our approach is based on monotone iteration, coupled with the concept of upper and lower solutions. In this section, appropriate upper and lower solutions are given. First of all, we define the characteristic function of (1.5) at \( x^0 = 0 \) by
\[ \Delta(\lambda, c, x^0) = -c\lambda + 1 - \mu \alpha \int_0^\tau K_1(u)e^{\lambda cu} \, du - \mu \beta \int_0^\tau K_2(u)e^{\lambda(1+cu)} \, du. \] (2.1)

The following lemma is to describe the property of (2.1) under the assumption \((H_f)\), which plays crucial roles in our study.

**Lemma 2.1.** Assume that \(\alpha \geq 1\). There exist exactly a pair of numbers \((c_*, \lambda_*)\) with \(c_* < 0, \lambda_* = \lambda(c_*) > 0\) such that

(i) \(\Delta(\lambda_*, c_*, x^0) = 0, \frac{\partial}{\partial \lambda} \Delta(\lambda_*, c_*, x^0) = 0;\)

(ii) for \(c_* < c \leq 0\), \(\Delta(\lambda, c, x^0) < 0\) for any \(\lambda \in \mathbb{R}\);

(iii) for any \(c < c_*\), there exist \(\lambda_1 > 0, \varepsilon_1 > 0\) such that \(\Delta(\lambda_1, c, x^0) = 0\), and for any small \(\varepsilon \in (0, \varepsilon_1)\) one has \(\Delta(\lambda_1 + \varepsilon, c, x^0) > 0\).

**Proof.** By a simple calculation we can obtain

\[
\begin{align*}
\frac{\partial}{\partial c} \Delta(\lambda, c, x^0) &= -\lambda \left(1 + \mu \alpha \int_0^\tau K_1(u)e^{\lambda cu} \, du + \mu \beta \int_0^\tau K_2(u)e^{\lambda(1+cu)} \, du\right), \\
\frac{\partial}{\partial \lambda} \Delta(\lambda, c, x^0) &= -c - \mu \alpha c \int_0^\tau K_1(u)e^{\lambda cu} \, du - \mu \beta \int_0^\tau K_2(u)(1 + cu)e^{\lambda(1+cu)} \, du.
\end{align*}
\]

Then, if \(c = 0\),

\[
\lim_{\lambda \to \infty} \frac{\partial}{\partial \lambda} \Delta(\lambda, c, x^0) = \lim_{\lambda \to \infty} -\mu \beta e^\lambda = -\infty.
\]

If \(c < 0\),

\[
\begin{align*}
\lim_{\lambda \to \infty} \int_0^\tau K_1(u)e^{\lambda cu} \, du &= 0, \\
\lim_{\lambda \to \infty} \int_0^{-\frac{1}{2}} K_2(u)(1 + cu)e^{\lambda(1+cu)} \, du &= \infty, \\
\lim_{\lambda \to \infty} \int_{-\frac{1}{2}}^\tau K_2(u)(1 + cu)e^{\lambda(1+cu)} \, du &= 0.
\end{align*}
\]

Thus, one also has

\[
\lim_{\lambda \to \infty} \frac{\partial}{\partial \lambda} \Delta(\lambda, c, x^0) = -\infty.
\]

Meanwhile, \(\Delta(\lambda, c, x^0)\) is a concave function of \(\lambda \in \mathbb{R}\) for any given \(c \in \mathbb{R}\) since

\[
\frac{\partial^2}{\partial \lambda^2} \Delta(\lambda, c, x^0) = -\mu \alpha c^2 \int_0^\tau K_1(u)u^2e^{\lambda cu} \, du - \mu \beta \int_0^\tau K_2(u)(1 + cu)^2e^{\lambda(1+cu)} \, du < 0.
\]

Note that,

\[
\Delta(0, c, x^0) = 1 - \mu(\alpha + \beta) < 0 \quad \text{for any } c \in \mathbb{R},
\]

\[
\lim_{c \to -\infty} \Delta(\lambda, c, x^0) = +\infty \quad \text{for any } \lambda > 0,
\]

\[
\Delta(\lambda, 0, x^0) = 1 - \mu \alpha - \mu \beta e^\lambda < 0 \quad \text{for any } \lambda \in \mathbb{R}.
\]

Therefore, there exist exactly a pair of numbers \((c_*, \lambda_*)\) with \(c_* < 0, \lambda_* = \lambda(c_*) > 0\) satisfying (i) and (iii).

Note that \(\Delta(\lambda, 0, x^0) < 0\) for any \(\lambda \in \mathbb{R}\). Furthermore, for any given \(\lambda < 0\), we have \(\frac{\partial \Delta}{\partial c} > 0\). Therefore \(\Delta(\lambda, c, x^0) = 0\) has no real roots for any \(c \in (c_*, 0]\). This completes the proof. \(\blacksquare\)

The concept of upper and lower solutions are introduced below.
Definition 2.1. A function \( V : \mathbb{R} \to \mathbb{R} \) is called an upper solution of (1.5) if it is differentiable almost everywhere (a.e.) and satisfies the inequality
\[
-cV'(s) \geq -V(s) + \alpha \int_0^s K_1(u) f(V(s + cu))du + \beta \int_0^s K_2(u) f(V(s + 1 + cu))du.
\]
Similarly, a function \( v : \mathbb{R} \to \mathbb{R} \) is called a lower solution of (1.5) if it is differentiable almost everywhere and satisfies the inequality
\[
-cv'(s) \leq -v(s) + \alpha \int_0^s K_1(u) f(V(s + cu))du + \beta \int_0^s K_2(u) f(v(s + 1 + cu))du.
\]
Appropriate upper and lower solutions are given now. First we consider the case of \( c < c_+ \). Define two functions as following:
\[
V(s) = \begin{cases} 
x^+ & s \geq 0, 
x^+e^{\lambda_1 s} & s < 0,
\end{cases}
\]
and
\[
v(s) = \begin{cases} 
0 & s \geq 0, 
\eta(1 - e^{\varepsilon s})e^{\lambda_1 s} & s < 0,
\end{cases}
\]
where \( x^+ \) is defined in (1.7), \( \lambda_1, \varepsilon \) are as in Lemma 2.1 and \( \eta \in (0, 1) \) is chosen small so that \( V(s) \geq v(s) \) and to be decided in the following such that \( v(s) \) is a lower solution. Clearly, we have \( 0 \leq v(s) \leq V(s) \leq x^+ \) and \( v(s) \neq 0 \) for \( s \in \mathbb{R} \).

Lemma 2.2. For any \( c < c_+ < 0 \), \( V \) is an upper solution and \( v \) is a lower solution of (1.5).

Proof. If \( s \geq 0 \), \( V(s) = x^+ \). Note that \( f(u) \leq 1 \) for any \( u \in \mathbb{R} \), then we have
\[
cV'(s) - V(s) + \alpha \int_0^s K_1(u) f(V(s + cu))du + \beta \int_0^s K_2(u) f(V(s + 1 + cu))du \leq 0 - x^+ + \alpha + \beta = 0.
\]
If \( s \leq 0 \), \( V(s) = x^+e^{\lambda_1 s} \). Note that \( V(s) \leq x^+e^{\lambda_1 s} \) for \( s \in \mathbb{R} \). Therefore, if \( x^+e^{\lambda_1 s} > 1 \), one has \( f(V(s)) = 1 < x^+e^{\lambda_1 s} \leq \mu x^+e^{\lambda_1 s} \); if \( x^+e^{\lambda_1 s} \leq 1 \), one also has \( f(V(s)) \leq \mu V(s) = \mu x^+e^{\lambda_1 s} \) from the assumption (3) in (H_{f}). According to Lemma 2.1, we have
\[
cV'(s) - V(s) + \alpha \int_0^s K_1(u) f(V(s + cu))du + \beta \int_0^s K_2(u) f(V(s + 1 + cu))du \\
\leq \ c\lambda_1 x^+e^{\lambda_1 s} - x^+e^{\lambda_1 s} + \mu \alpha \int_0^s K_1(u)x^+e^{\lambda_1(s + cu)}du + \mu \beta \int_0^s K_2(u)x^+e^{\lambda_1(s + 1 + cu)}du \\
\quad = -x^+e^{\lambda_1 s} \Delta(\lambda_1, c, x^0) = 0.
\]
So \( V(s) \) is an upper solution of (1.5).

Next we show that \( v \) is a lower solution of (1.5). That is,
\[
cv'(s) - v(s) + \alpha \int_0^s K_1(u) f(v(s + cu))du + \beta \int_0^s K_2(u) f(v(s + 1 + cu))du \geq 0.
\]
If \( s \geq 0 \), \( v'(s) = v(s) \equiv 0 \). For the last two integral terms of the left-hand side of (2.4), we distinguish the following three cases, namely, \( s \geq -c\tau \), \( -c\tau - 1 \leq s \leq -c\tau \) and \( s < -c\tau - 1 \).

Case 1. \( s \geq -c\tau \). In this case for \( u \in [0, \tau] \) we have \( s + 1 + cu > s + cu \geq s + c\tau \geq 0 \), hence \( f(v(s + cu)) = f(v(s + 1 + cu)) \equiv 0 \), (2.4) holds obviously.

Case 2. \( -c\tau - 1 \leq s \leq -c\tau \). In this case for \( u \in [0, \tau] \) we have \( s + 1 + cu \geq s + 1 + c\tau \geq 0 \), and hence \( f(v(s + 1 + cu)) \equiv 0 \). Furthermore
\[
\int_0^\tau K_1(u) f(v(s + cu))du = \int_{-\frac{\tau}{c}}^0 K_1(u) f(v(s + cu))du + \int_{\frac{\tau}{c}}^\tau K_1(u) f(v(s + cu))du
\]
Thus, (2.4) holds in this case.

**Case 3.** \( s < -c\tau -1 \). In this case, for \( u \in [0, -\frac{s}{c}] \), \( s + cu \geq 0 \), \( f(v(s + cu)) \equiv 0 \), and for \( u \in \left[0, -\frac{s + 1}{c}\right] \), \( s + 1 + cu \geq 0 \), \( f(v(s + 1 + cu)) \equiv 0 \). So we get that

\[
\int_0^s K_1(u)f(v(s + cu))du = \int_{\gamma}^{\frac{s}{c}} K_1(u)f(v(s + cu))du + \int_{\frac{s}{c}}^\gamma K_1(u)f(v(s + cu))du \\
= 0 + \int_{\gamma}^{\frac{s}{c}} f(\eta(1 - e^{e(s+cu)})e^{\lambda_1(s+cu)})du \geq 0.
\]

Thus, (2.4) still holds in this case.

If \( s < 0 \), there are also three cases, namely, \(-(c\tau + 1) < s < -c\tau \), \(-1 < s \leq -(c\tau + 1)\), and \( s \leq -1 \).

**Case 1.** \(-(c\tau + 1) < s < 0\). In this case, for \( u \in [0, \tau] \), one has \( s + 1 + cu > 0 \), which implies \( v(s + 1 + cu) = 0 \).

Moreover, if \( s < 0, 0 < \eta < 1 \), one has \( 0 < \eta e^{\lambda_1(s+cu)}(1 - e^{e(s+cu)}) < 1 \). So using (1.9) and (2.3) we can obtain

\[
cv'(s) - v(s) + \alpha \int_0^\tau K_1(u)f(v(s + cu))du + \beta \int_0^\tau K_2(u)f(v(s + 1 + cu))du \\
= -\eta e^{\lambda_1}(-c\lambda_1 + 1) + \eta e^{(\lambda_1+\varepsilon)s} \frac{c}{1 + cu} \lambda_1(1 - e^{e(s+cu)})e^{\lambda_1(s+cu)}du \\
- \alpha|\sigma(\eta)| \int_0^\tau K_1(u)(1 - e^{e(s+cu)})e^{\lambda_1(s+cu)}du \\
= -\eta e^{\lambda_1} \Delta(\lambda_1, c, x^0) + \eta e^{(\lambda_1+\varepsilon)s} \Delta(\lambda_1 + \varepsilon, c, x^0) + G_1,
\]

where

\[
G_1 = -\mu \beta \eta \int_0^\tau K_2(u)e^{\lambda_1(s+1+cu)}[1 - e^{e(s+1+cu)}]du - \alpha|\sigma(\eta)| \int_0^\tau K_1(u)(1 - e^{e(s+cu)})e^{\lambda_1(s+cu)}du \\
\geq \eta \left(\mu \beta \eta \int_0^\tau K_2(u)e^{\lambda_1(s+1+cu)}[1 - e^{e(s+1+cu)}]du - \alpha|\sigma(\eta)| \int_0^\tau K_1(u)(1 - e^{e(s+cu)})e^{\lambda_1(s+cu)}du \right).
\]

Thus, \( G_1 \geq 0 \) if \( \eta \) is small enough. From Lemma 2.1, \( \Delta(\lambda_1, c, x^0) = 0 \), \( \Delta(\lambda_1 + \varepsilon, c, x^0) > 0 \), hence (2.4) holds.

**Case 2.** \(-1 < s \leq -(c\tau + 1) \). In this case, for \( u \in [0, -\frac{s+1}{c}] \), one has \( s + 1 + cu \geq 0 \), and for \( u \in (-\frac{s+1}{c}, \tau] \) one has \( s + 1 + cu < 0 \). Choose \( \eta \in (0, 1) \), we have

\[
cv'(s) - v(s) + \alpha \int_0^\tau K_1(u)f(v(s + cu))du + \beta \int_0^\tau K_2(u)f(v(s + 1 + cu))du \\
= -\eta e^{\lambda_1}(-c\lambda_1 + 1) + \eta e^{(\lambda_1+\varepsilon)s} \frac{c}{1 + cu} \lambda_1(1 - e^{e(s+cu)})e^{\lambda_1(s+cu)}du \\
- \alpha|\sigma(\eta)| \int_0^\tau K_1(u)(1 - e^{e(s+cu)})e^{\lambda_1(s+cu)}du + \beta \int_{\frac{s}{c}}^\gamma K_2(u)f(\eta(1 - e^{e(s+1+cu)})e^{\lambda_1(s+1+cu)}du \\
= -\eta e^{\lambda_1} \Delta(\lambda_1, c, x^0) + \eta e^{(\lambda_1+\varepsilon)s} \Delta(\lambda_1 + \varepsilon, c, x^0) + G_2 \\
= \eta e^{(\lambda_1+\varepsilon)s} \Delta(\lambda_1 + \varepsilon, c, x^0) + G_2,
\]

where \( G_2 = G_{21} + G_{22} + G_{23} \) and

\[
G_{21} = -\alpha|\sigma(\eta)| \int_0^\tau K_1(u)(1 - e^{e(s+cu)})e^{\lambda_1(s+cu)}du,
\]
\[ G_{22} = -\mu \beta \eta \int_{0}^{\tau} K_2(u)e^{\hat{\lambda}_1(s+cu)}[1 - e^{\epsilon(s+cu)}]du, \]
\[ G_{23} = \beta (\mu \eta - |\sigma(\eta)|) \int_{s+1}^{\tau} K_2(u)e^{\hat{\lambda}_1(s+cu)}(1 - e^{\epsilon(s+cu)})du. \]

Note that
\[ G_{22} + G_{23} = \mu \beta \eta \int_{0}^{\tau} K_2(u)e^{\hat{\lambda}_1(s+cu)}[e^{\epsilon(s+cu)} - 1]du \]
\[ - \beta |\sigma(\eta)| \int_{s+1}^{\tau} K_2(u)e^{\hat{\lambda}_1(s+cu)}(1 - e^{\epsilon(s+cu)})du \]
\[ \geq \mu \beta \eta \int_{0}^{\tau} K_2(u)e^{\hat{\lambda}_1(s+cu)}[e^{\epsilon(s+cu)} - 1]du \]
\[ - \beta |\sigma(\eta)| \int_{s+1}^{\tau} K_2(u)e^{\hat{\lambda}_1(s+cu)}(1 - e^{\epsilon(s+cu)})du \]
\[ \geq \mu \beta \eta \left[ e^{\hat{\lambda}_1(c\epsilon - 1)} \int_{0}^{\tau} K_2(u)du - \frac{|\sigma(\eta)|}{\eta} \int_{s+1}^{\tau} K_2(u)e^{\hat{\lambda}_1(s+cu)}(1 - e^{\epsilon(s+cu)})du \right]. \]

Therefore \( G_2 \geq 0 \) if \( \eta \) is small enough, and similar to case 1, we have (2.4).

Case 3. \( s \leq -1 \). In this case for any \( u \in [0, \tau] \), \( s + 1 + cu \leq 0 \). Hence
\[ cu'(s) - v(s) + \alpha \int_{0}^{\tau} K_1(u)f(v(s+cu))du + \beta \int_{0}^{\tau} K_2(u)f(v(s+1+cu))du \]
\[ = -\eta e^{\hat{\lambda}_1(s)}(-c\lambda_1 + 1) + \eta e^{(\lambda_1+\epsilon)s}[-c(\lambda_1 + \epsilon) + 1] \]
\[ + \alpha \int_{0}^{\tau} K_1(u)f(\eta e^{\hat{\lambda}_1(s+cu)}(1 - e^{\epsilon(s+cu)}))du + \beta \int_{0}^{\tau} K_2(u)f(\eta e^{\hat{\lambda}_1(s+1+cu)}(1 - e^{\epsilon(s+1+cu)}))du \]
\[ = -\eta e^{\hat{\lambda}_1(s)}\Delta(\lambda_1, c, x^0) + \eta e^{(\lambda_1+\epsilon)s}\Delta(\lambda_1 + \epsilon, c, x^0) + G_3 \]
\[ = \eta e^{(\lambda_1+\epsilon)s}\Delta(\lambda_1 + \epsilon, c, x^0) + G_3, \]
where \( G_3 = G_{31} + G_{32} \) and
\[ G_{31} = -\alpha |\sigma(\eta)| \int_{0}^{\tau} K_1(u)e^{\hat{\lambda}_1(s+cu)}(1 - e^{\epsilon(s+cu)})du \]
\[ G_{32} = -\beta |\sigma(\eta)| \int_{0}^{\tau} K_2(u)e^{\hat{\lambda}_1(s+1+cu)}(1 - e^{\epsilon(s+1+cu)})du. \]

So we can choose \( \eta \) small enough such that
\[ \eta e^{(\lambda_1+\epsilon)s}\Delta(\lambda_1 + \epsilon, c, x^0) + G_3 > 0. \]

Hence (2.4) still holds in this case.

According to the above discussion, we know that \( v(s) \) is a lower solution of (1.5). This completes the proof. \( \blacksquare \)

3. Main results

In this section we will give the existence of traveling solutions of (1.1). The case of \( c < c_u < 0 \) is first considered. Let \( \mathcal{C} = \mathcal{C}(\mathbb{R}, [x^0, x^+]) \), and
\[ S_1 = \left\{ \phi \in \mathcal{C} : \begin{array}{ll}
\text{(i) } & \phi(s) \text{ is nondecreasing for } s \in \mathbb{R}; \\
\text{(ii) } & \lim_{s \to -\infty} \phi(s) = x^0, \lim_{s \to \infty} \phi(s) = x^+. \end{array} \right\}. \]

Consider the following equivalent form of Eq. (1.5):
\[ \frac{d\phi(s)}{ds} + \gamma \phi(s) = F(\phi(s)), \]
(3.2)
For any c

The proof of (1) is a direct verification by L’Hospital rule. On the other hand, the monotonicity of

if

if

(2.2)

which is increasing and satisfying

\( (1.1) \)

\( (1.1) \)

Proposition 3.1

Lemma 2.2

satisfies

if

Assume that c

\( (3.2) \)

Theorem 3.1.

Proof. Proposition 3.1.

\[ \phi(s) = e^{-\gamma s} \int_{-\infty}^{s} e^{\nu u} F(\phi(u))du. \]  

(3.3)

Define an operator \( T: S_1 \rightarrow C \) by

\[ (T \phi)(s) = e^{-\gamma s} \int_{-\infty}^{s} e^{\nu u} F(\phi(u))du, \quad \phi \in S_1, s \in \mathbb{R}. \]  

(3.4)

Then we have the following

**Proposition 3.1.** Assume that \( c < c_* < 0 \). Then \( T \) defined in (3.4) satisfies

1. If \( \phi \in S_1 \), then \( T \phi \in S_1 \);
2. If \( \phi \) is an upper (a lower) solution of (1.5), then \( \phi(s) \geq (T \phi)(s) \) (\( \phi(s) \leq (T \phi)(s) \)) for \( s \in \mathbb{R} \);
3. If \( \phi(s) \geq \psi(s) \) for \( s \in \mathbb{R} \), then \( (T \phi)(s) \geq (T \psi)(s) \) for \( s \in \mathbb{R} \);
4. If \( \phi \) is an upper (a lower) solution of (1.5), then \( T \phi \) is also an upper (a lower) solution of (1.5).

**Proof.** The proof of (1) is a direct verification by L’Hospital rule. On the other hand, the monotonicity of \( F \) leads to the conclusion (3). In the following, we only show that (2) and (4) hold. In fact, if \( \phi(s) \) is an upper solution of (1.5), then

\[ \frac{d \phi(s)}{ds} + \gamma \phi(s) \geq F(\phi(s)). \]

This leads to

\[ \frac{d(e^{\gamma s} \phi(s))}{ds} \geq e^{\gamma s} F(\phi(s)). \]

Integrating the inequality above from \(-\infty\) to \( s \), we obtain (2).

Noting that \( F(\phi)(s) \geq F((T \phi)(s)) \) from (2), we have

\[ \frac{d(T \phi)(s)}{ds} + \gamma (T \phi)(s) = F(\phi)(s) \geq F((T \phi)(s)). \]

This means that \( (T \phi)(s) \) is also an upper solution of (1.5). The proof is similar if \( \phi(s) \) is a lower solution. Thus the proof is complete. ■

Consider the following iterative scheme:

\[ V_0 = V \quad \text{and} \quad V_n = TV_{n-1}, \quad n = 1, 2, \ldots, \]

where \( V \) is defined in (2.2). By Lemma 2.2 and Proposition 3.1, we have

\[ x^0 \leq v(s) \leq V_n(s) \leq V_{n-1}(s) \leq \cdots \leq V(s) \leq x^+. \]

By Lebesgue’s dominated convergence theorem, the limit function \( \phi(s) = \lim_{n \to \infty} V_n(s) \geq v(s) \) exists and is a fixed point of \( T \). Therefore \( \phi \) is a solution of (1.1) and satisfies

\[ \lim_{s \to -\infty} \phi(s) = x^0, \quad \lim_{s \to \infty} \phi(s) = x^+. \]

Thus, \( \phi(s) \) is a monotone traveling wave solution and we obtain the following theorem.

**Theorem 3.1.** For any \( c < c_* < 0 \), there exists a wave solution \( \phi(s) \) of (1.1) which is increasing and satisfying

\[ \lim_{s \to -\infty} \phi(s) = x^0, \quad \lim_{s \to \infty} \phi(s) = x^+. \]
Note that $f$ is an odd function, let $\psi = -\phi$, then (1.5) is changed to
\begin{equation}
-c\psi'(s) = -\psi(s) + \alpha \int_0^s K_1(u) f(\psi(u))\,du + \beta \int_0^s K_2(u) f(\psi(u + 1))\,du
\end{equation}
which is exactly of the same form as (1.5), so we have the following.

**Theorem 3.2.** For any $c < c_* < 0$, there exists a wave solution $\phi(s)$ of (1.1) which is decreasing and satisfying
\begin{align*}
\lim_{s \to -\infty} \phi(s) &= x^0, \\
\lim_{s \to \infty} \phi(s) &= x^-. 
\end{align*}

In the rest of this section, we will discuss the existence of monotone traveling waves of (1.1) for the case of $c > 0$. By the facts
\begin{align*}
\Delta(0, c, x^0) &= 1 - \mu(\alpha + \beta) < 0; \\
\lim_{\lambda \to -\infty} \Delta(\lambda, c, x^0) &= +\infty; \\
\frac{\partial \Delta}{\partial \lambda}(\lambda, c, x^0) &= 0, \quad \lambda \in \mathbb{R}; \\
\frac{\partial^2 \Delta}{\partial \lambda^2}(\lambda, c, x^0) &= 0, \quad \lambda \in \mathbb{R}.
\end{align*}
Thus, we know, for any fixed $c > 0$, the equation $\Delta(\lambda, c, x^0) = 0$ has a unique real root $\lambda_2 = \lambda_2(c) < 0$. Furthermore, there is $\varepsilon_2 > 0$ such that for $0 < \varepsilon < \varepsilon_2$, one has
\begin{equation}
\Delta(\lambda_2 - \varepsilon, c, x^0) > 0.
\end{equation}

We note that (1.5) becomes
\begin{equation}
\phi'(s) = \frac{1}{c}(\phi(s) - \alpha - \beta)
\end{equation}
if $\phi(s) \geq 1$ for large $|s|$, and
\[\phi(s) = (\phi(0) - \alpha - \beta)e^s + \alpha + \beta.\]
Therefore, (3.7) has no monotone solution satisfying (1.8), so we consider monotone solutions with boundary conditions
\begin{align*}
\lim_{s \to -\infty} \phi(s) &= x^+, \\
\lim_{s \to \infty} \phi(s) &= x^0.
\end{align*}
Let
\[S_2 = \left\{ \phi \in C : \begin{array}{l}
(i) \phi(s) \text{ is nonincreasing for } s \in \mathbb{R}; \\
(ii) \lim_{s \to -\infty} \phi(s) = x^+, \lim_{s \to \infty} \phi(s) = x^0.
\end{array} \right\},\]
\[\bar{V}(s) = \begin{cases} x^+, & s \leq 0, \\
x^+e^{\lambda_2 s}, & s \geq 0, \end{cases} \]
and
\[\bar{v}(s) = \begin{cases} 0, & s \leq 0, \\
\eta(1 - e^{-\varepsilon s})e^{\lambda_2 s}, & s \geq 0, \end{cases} \]
where $\lambda_2, \varepsilon$ are given in (3.6). We can show that $\bar{V}(s)$ is an upper solution and $\bar{v}(s)$ is a lower solution of (1.5) while $\eta$ is appropriately chosen, with the argument similar to that for the situation where $c < c_* < 0$.

Let
\[H(\phi)(s) = \left(\frac{1}{c} - \gamma\right)\phi(s) - \frac{\alpha}{c} \int_0^s K_1(u) f(\phi(u))\,du - \frac{\beta}{c} \int_0^s K_2(u) f(\phi(u + 1))\,du.\]
We choose $\gamma > \frac{1}{c}$ such that $H(\phi)(s) \leq H(\psi)(s)$ provided that $\phi(s) \geq \psi(s)$ for $s \in \mathbb{R}$. Note that (1.5) is equivalent to
\[
\phi(s) = -e^{\gamma s} \int_s^\infty e^{-\gamma u} H(\phi)(u) du.
\] (3.9)
Define an operator $Q : S_2 \to \mathbb{C}$ by
\[
(Q\phi)(s) = -e^{\gamma s} \int_s^\infty e^{-\gamma u} H(\phi)(u) du, \quad \phi \in S_2, \quad s \in \mathbb{R}.
\] (3.10)
Similar to the proof of Proposition 3.1 we have

**Proposition 3.2.** Let $Q$ be defined in (3.10). Then

1. if $\phi \in S_2$, then $Q\phi \in S_2$;
2. if $\phi$ is an upper (a lower) solution of (1.5), then $(Q\phi)(s) < (Q\psi)(s)$ for $s \in \mathbb{R}$;
3. if $\phi(s) \geq \psi(s)$ for $s \in \mathbb{R}$, then $(Q\phi)(s) \geq (Q\psi)(s)$ for $s \in \mathbb{R}$;
4. if $\phi$ is an upper (a lower) solution of (1.5), then $Q\phi$ is also an upper (a lower) solution of (1.5).

Then we can show the existence of a monotone solution in $S_2$ of (1.5) by monotone iteration method, with the argument similar to that of the situation where $c < c^* < 0$. In particular, we have the following.

**Theorem 3.3.** For any $c > 0$, we have the following conclusions.

1. There exists a wave solution $\phi(s)$ of (1.1) which is decreasing and satisfying
   \[
   \lim_{s \to -\infty} \phi(s) = x^+, \quad \lim_{s \to \infty} \phi(s) = x^0.
   \]
2. There exists a wave solution $\phi(s)$ of (1.1) which is increasing and satisfying
   \[
   \lim_{s \to -\infty} \phi(s) = x^-, \quad \lim_{s \to \infty} \phi(s) = x^0.
   \]

Finally, we shall briefly discuss the existence of monotone waves of CNN model with some explicit output function $f$.

**Example 3.1.** Let the output function $f$ be defined as (1.3). Obviously, the assumption $(H_f)$ is satisfied with $\mu = 1$. Then (1.1) has monotone waves by Theorems 3.1–3.3, which leads to Theorems 4.1–4.2 in [10].

**Example 3.2.** Let the output function $f$ be defined by
\[
f(x) = \begin{cases} 1 & \text{if } x \geq 1, \\ \sin \frac{\pi}{2} x & \text{if } -1 \leq x \leq 1, \\ -1 & \text{if } x \leq -1. \end{cases}
\] (3.11)
Then the assumption $(H_f)$ is satisfied with $\mu = \frac{\pi}{2} > 1$. Hence (1.1) has monotone waves by Theorems 3.1–3.3.

**Example 3.3.** Let the output function $f$ be defined by
\[
f(x) = \begin{cases} 1 & \text{if } x \geq 1, \\ 2x - x^2 & \text{if } 0 \leq x \leq 1, \\ 2x + x^2 & \text{if } -1 \leq x \leq 0, \\ -1 & \text{if } x \leq -1. \end{cases}
\] (3.12)
Obviously the assumption $(H_f)$ is satisfied with $\mu = 2$, which implies that (1.1) has monotone waves by Theorems 3.1–3.3.
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