Periodic Boundary Value Problems and Periodic Solutions of Second Order FDE with Upper and Lower Solutions

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Abstract. We use the monotone iterative technique with upper and lower solutions in reversed order to obtain two monotone sequences that converge uniformly to extremal solutions of second order periodic boundary value problems and periodic solutions of functional differential equations (FDEs).

Keywords: Periodic boundary value problem; Periodic solution; Existence; Upper and lower solutions; Monotone iterative technique.

1. Introduction

The method of upper and lower solutions coupled with the monotone iterative technique has been applied successfully to obtain results of the existence and approximation of solutions for periodic boundary value problems of first order and second order ordinary differential equations (see [1-2] and references therein) and first order functional differential equations (see [3-7] and references therein). However, as far as the authors know, the method of upper and lower solutions coupled with the monotone iterative technique has rarely been seen for periodic boundary value problems and periodic solutions of second functional differential equations, and only in [8] Jiang ang Wei have dealt with the second order

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functional differential equations under the classical assumption that \( \alpha(t) \leq \beta(t) \).

We consider the following periodic boundary value problem (PBVP)

\[
\begin{aligned}
    y''(t) &= f(t, y(t), y(w(t))), \quad t \in I := [0, T], \\
    y(0) &= y(T), \quad y'(0) = y'(T);
\end{aligned}
\]  

where \( f \in C(I \times \mathbb{R}^2, \mathbb{R}) \), \( w \in C(I, [a, b]) \), and \( a, b \) are constants such that \([0, T] \subset [a, b], T > 0\).

Similarly, we shall also consider the \( T \)-periodic solutions of the following functional differential equation (FDE):

\[
    y''(t) = f(t, y(t), y(w(t))), \quad t \in \mathbb{R};
\]

where \( f \in C(R^3, \mathbb{R}), f(t, u, v) = f(t + T, u, v), T > 0, \ w(t) = t - \tau(t), \ \tau \in C(R, R), \ \tau(t) = \tau(t + T). \)

Suppose that \( \beta(t) \) and \( \alpha(t) \) are upper and lower solutions respectively. For the second order ordinary differential equations, only a few of people have dealt with the case \( \beta(t) \leq \alpha(t) \) (for example, see [9-13] and references therein). If \( \beta(t) \leq \alpha(t) \), then the monotone method is not valid in general. Consider, for example, the following problem

\[
    u'' + u = \cos t + \sin t, \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),
\]

which has no solution although \( \alpha(t) = 2 \) and \( \beta(t) = -2 \) are lower and upper solutions. It is proved in [10] that the monotone method is valid if \( f(t, u, v) \equiv f(t, u) \) is a continuous function which satisfies an one-side Lipschitz condition:

\[
    \text{for } t \in [0, T], \text{ and all } u, v \in [\beta(t), \alpha(t)],
    \quad u \leq v \Rightarrow f(t, u) + Mu \leq f(t, v) + Mv,
\]

here \( M \) is a positive constant such that the operator \( u'' + Mu \) is inverse positive in the space

\[
    F_2 = \{ u \in C^2[0, T] \mid u(0) = u(T), \quad u'(0) \geq u'(T) \}.
\]

Note that \( u'' + Mu \) is inverse positive in \( F_2 \) if and only if \( M \in (0, (\pi^2)^2) \) (see [10]).

The main purpose of this paper is to deal with (1.1) and (1.2) with upper and lower solutions in the reversed order \( \beta(t) \leq \alpha(t) \). The method that we used for dealing with anti-maximum principle is different from [8].

In this paper, Section 2 is devoted to the anti-maximum principle which is the key to develop the monotone technique. Section 3 is devoted to the monotone method for (1.1) and (1.2) with upper and lower solutions in the reversed order.

2. Anti-maximum Principle
In the proof of our anti-maximum principle, we adopt a lemma from Cabada in [10].

**Lemma 2.1.** If the linear problem
\[
\begin{align*}
  z^{(n)}(t) + Mz(t) &= 0, \quad t \in [a, b], \\
  z^{(i)}(a) - z^{(i)}(b) &= 0, \quad i = 0, \ldots, n - 2, \\
  z^{(n-1)}(a) - z^{(n-1)}(b) &= 1
\end{align*}
\]
has a unique solution \( r \in C^\infty[a, b] \), then the problem
\[
\begin{align*}
  u^{(n)}(t) + Mu(t) &= \sigma(t) \in L^1[a, b], \\
  u^{(i)}(a) - u^{(i)}(b) &= \lambda_i, \quad i = 0, \ldots, n - 1,
\end{align*}
\]
has a unique solution \( u \) given by the expression
\[
u(t) = \int_a^b G_n(t, s)\sigma(s)ds + \sum_{i=0}^{n-1} r^{(i)}(t)\lambda_{n-1-i},
\]
where
\[
G_n(t, s) = \begin{cases} r(a + t - s), & a \leq s \leq t \leq b \\ r(b + t - s), & a \leq t \leq s \leq b. \end{cases}
\]

**Theorem 2.2.** Let \( y \in S = C([a, b], R) \cap C^2(I, R) \) and \( M \in (0, (\frac{\pi}{2})^2) \), \( N > 0 \) such that
(i) \( y''(t) + My(t) + Ny(w(t)) \geq 0, \quad t \in I; \)
(ii) \( y(0) = y(T), \quad y'(0) \geq y'(T); \)
(iii) \( y(0) = y(t), \quad t \in [a, 0] \cup [T, b]; \)
(iv) \( \frac{N}{M} \sec \frac{\sqrt{MT}}{2} - 1 < 1, \) where \( w \in C(I, [a, b]). \)
Then, \( y(t) \geq 0, \) for all \( t \in [a, b]. \)

**Proof.** There exists \( \xi \in [0, T] \) such that
\[
y(\xi) = \max_{t \in [0, T]} y(t) = \max_{t \in [a, b]} y(t),
\]
thus
\[
y''(t) + My(t) + Ny(\xi) \geq 0, \quad t \in I,
\]
\[
y(0) = y(T), \quad y'(0) \geq y'(T).
\]
Let \( \sigma(t) = y''(t) + My(t) + Ny(\xi) \geq 0 \) and \( \lambda = y'(0) - y'(T) \geq 0. \) Then we have
\[
y''(t) + My(t) + Ny(\xi) = \sigma(t), \quad t \in I,
\]
\[
y(0) = y(T), \quad y'(0) = y'(T) + \lambda.
\]
Let \( u(t) = y(t) + \frac{N}{M} y(\xi) \). Then we have

\[
\begin{align*}
    u''(t) + Mu(t) &= \sigma(t), \quad t \in I, \\
    u(0) &= u(T), \quad u'(0) = u'(T) + \lambda.
\end{align*}
\]

Let \( r(t) \) be the unique solution \( r \in C^\infty[0,T] \) to the problem

\[
\begin{align*}
    z''(t) + Mz(t) &= 0, \quad z(0) = z(T), \quad z'(0) = z'(T) + 1.
\end{align*}
\]

Then we have

\[
\begin{align*}
    r(t) &= \sin m(T - t) + \sin mt \frac{2m(1 - \cos mT)}{m(1 - \cos mT)},
\end{align*}
\]

By Lemma 2.1, we obtain

\[
\begin{align*}
    u(t) &= \left\{ \begin{array}{ll}
        \int_0^T G(t, s) \sigma(s) ds + \lambda r(t), & t \in [0, T], \\
        \int_0^T G(0, s) \sigma(s) ds + \lambda r(0), & t \in [a, 0] \cup [T, b],
    \end{array} \right. \quad (2.1)
\end{align*}
\]

where \( m = \sqrt{M} \), and

\[
G(t, s) := \left\{ \begin{array}{ll}
        \frac{\sin m(t-s) + \sin m(T-t+s)}{2m(1 - \cos mT)}, & 0 \leq s \leq t \leq T, \\
        \frac{\sin m(T+t-s) + \sin m(s-t)}{2m(1 - \cos mT)}, & 0 \leq t \leq s \leq T.
    \end{array} \right. \quad (2.2)
\]

A direct calculation shows that

\[
0 < \frac{\sin \frac{mtT}{2} \cos \frac{mtT}{2}}{m(1 - \cos mT)} = r(0) \leq G(t, s) \leq r\left( \frac{T}{2} \right) = \frac{\sin \frac{mtT}{2}}{m(1 - \cos mT)}. \quad (2.3)
\]

Thus, from (2.1) and (2.3), we have

\[
r(0) \left( \int_0^T \sigma(s) ds + \lambda \right) \leq u(t) \leq r\left( \frac{T}{2} \right) \left( \int_0^T \sigma(s) ds + \lambda \right), \quad \text{for all } t \in [a, b]. \quad (2.4)
\]

By (2.4), we obtain

\[
y(\xi) = \frac{M}{M + N} u(\xi) \leq \frac{M}{M + N} r\left( \frac{T}{2} \right) \left( \int_0^T \sigma(s) ds + \lambda \right).
\]

On the other hand, by (2.4) we also have

\[
y(t) = u(t) - \frac{N}{M} y(\xi)
\geq r(0)(\lambda + \int_0^T \sigma(s) ds) - \frac{N}{M} y(\xi)
\geq (\lambda + \int_0^T \sigma(s) ds) r\left( \frac{T}{2} \right) (\delta - \frac{N}{M + N}),
\]
here $\delta = \frac{r(0)}{r(2)} = \cos \frac{mT}{\sqrt{N}}$. From the assumption (iv), we obtain $\frac{N}{M+N} < \delta$. So we obtain that $y(t) \geq 0$, for all $t \in [a,b]$. \hfill \blacksquare

Similarly, we can obtain the following theorem, but we omit the details.

**Theorem 2.3.** Let $y \in X = \{ y \in C^2(R, R) : y(t) = y(t + T), \; T > 0 \}$ and $M \in (0, \left(\frac{\pi}{2}\right)^2), \; N > 0$ such that

(i) $y''(t) + My(t) + Ny(w(t)) \geq 0, \; t \in R$;

(ii) $\frac{N}{M} (\sec \frac{\pi T}{2} - 1) < 1$, where $w(t) = t - \tau(t), \; \tau \in C(R, R), \; \tau(t) = \tau(t + T)$. Then $y(t) \geq 0$, for all $t \in R$.

**3. Monotone Method of the Reversed Order**

We shall first consider the following boundary value problem for the linear equation:

$$
\begin{cases}
y''(t) + My(t) + Ny(w(t)) = \sigma(t), & t \in I, \\
y(0) = y(T), & y'(0) = y'(T), \\
y(t) = y(0), & t \in [a,0] \cup [T,b],
\end{cases}
$$

(3.1)

where $\sigma \in C(I, R)$, $w \in C([a,b])$, for all $t \in I$.

Let $$E = \{ y \in S : y(t) = y(0), \; \text{for all } t \in [a,0] \cup [T,b] \},$$

where $S$ is defined in Theorem 2.2, and the norm in $E$ is

$$||y||_2 = \max_{t \in [a,b]} |y(t)| + \max_{t \in [0,T]} |y'(t)| + \max_{t \in [0,T]} |y''(t)|.$$ 

It is clear that $E$ is a Banach space.

A function $\alpha \in E$ is said to be a lower solution of (3.1), if it satisfies

$$
\begin{cases}
\alpha''(t) + M\alpha(t) + N\alpha(w(t)) \geq \sigma(t), & t \in I \\
\alpha(0) = \alpha(T), & \alpha'(0) \geq \alpha'(T).
\end{cases}
$$

(3.2)

An upper solution $\beta \in E$ of (3.1) is defined analogously by reversing the inequalities above.

For $\alpha, \beta \in E$, we shall write $\beta \leq \alpha$ if $\beta(t) \leq \alpha(t)$ for $t \in [a,b]$, and we denote $[\beta, \alpha] = \{ y \in E : \beta \leq y \leq \alpha \}$.

**Theorem 3.1.** Suppose that there exists a lower solution $\alpha$ and an upper solution $\beta$ of (3.1) such that $\beta \leq \alpha$, and assume that $M \in (0, \left(\frac{\pi}{2}\right)^2), \; N > 0$ satisfying condition (iv) of Theorem 2.2. Then (3.1) has a unique solution $y \in [\beta, \alpha]$.

**Proof.** Consider the boundary value problem

$$
\begin{cases}
y''(t) + My(t) = -Np(t, y(w(t))) + \sigma(t), & t \in I = [0,T], \\
y(0) = y(T), & y'(0) = y'(T), \\
y(t) = y(0), & t \in [a,0] \cup [T,b],
\end{cases}
$$

(3.1)*
where
\[ p(t,x) = \begin{cases} 
\beta(t), & \text{if } x < \beta(t), \\
x, & \text{if } \beta(t) \leq x \leq \alpha(t), \\
\alpha(t), & \text{if } x > \alpha(t). 
\end{cases} \]

Now, we define an operator \( \Phi : E \to E \) by
\[
(\Phi y)(t) = \begin{cases} 
\int_0^T G(t,s)(-Np(s,y(s))) + \sigma(s)ds, & t \in I = [0,T], \\
\int_0^T G(0,s)(-Np(s,y(s))) + \sigma(s)ds, & t \in [a,0] \cup [T,b],
\end{cases}
\]

where \( G(t,s) \) is defined in (2.2). Then it can be easily shown that \( \Phi : E \to E \) is completely continuous.

Since \( -Np(t,y(w(t))) + \sigma(t) \) is bounded on \( I \), \( \Phi \) is bounded in \( E \). The existence of a fixed point \( y \) for the operator \( \Phi \) follows from the Schauder fixed point theorem. This means that (3.1)* has a solution \( y \in E \).

Now we show that \( y \in [\beta, \alpha] \). First, we prove that \( y \geq \beta \). Set \( u(t) = y(t) - \beta(t) \), for \( t \in [a,b] \). Since \( p(t,y(w(t))) - \beta(w(t)) \leq \max\{u(w(t)),0\}, t \in I \), by the definition of upper solution, we obtain that
\begin{enumerate}
  \item\( u''(t) + Mu(t) + N \max\{u(w(t)),0\} \geq 0 \), \( t \in I \);
  \item\( u(0) = u(T), \ u'(0) \geq u'(T) \);
  \item\( u(0) = u(t), \ t \in [a,0] \cup [T,b] \);
  \item\( \frac{N}{M}(\sec \frac{\pi T}{2} - 1) < 1 \).
\end{enumerate}

Suppose, on the contrary, that \( y(t) < \beta(t) \), for some \( t \in [0,T] \). Then it suffices to consider the following two cases.

**Case 1:** \( u(t) \leq 0 \), \( u(t) \not= 0 \) on \( I = [0,T] \). In this case, we have \( u'(0) \geq u'(T), \ u''(t) \geq 0 \), \( t \in I \). Thus \( u(t) = \text{const.} = K < 0 \) on \( I \), and we obtain
\[ 0 \leq u''(t) + Mu(t) + N \max\{u(w(t)),0\} = MK, \]

which contradicts \( K < 0 \).

**Case 2:** There exist \( t_1, t_2 \in [0,T] \) such that \( u(t_1) > 0 \) and \( u(t_2) < 0 \). Define \( u(\xi) = \max_{t \in [0,T]} u(t) = \max_{t \in [a,b]} u(t) \). Since \( \max\{u(w(t)),0\} \leq u(\xi) \), we have
\[ u''(t) + Mu(t) + Nu(\xi) \geq 0, \ t \in [0,T]. \]

Similar to the proof of Theorem 2.2, we obtain \( u(t) \geq 0 \). This implies that \( y \geq \beta \).

Similarly, we can prove \( y \leq \alpha \). Therefore, we have \( y \in [\beta, \alpha] \). This implies that \( y \) is also a solution of (3.1).

Finally, we prove the uniqueness. Suppose that there exist two solutions \( y_1 \) and \( y_2 \) of (3.1) on \([\beta, \alpha] \). Applying Theorem 2.2 again, one can prove \( v = y_1 - y_2 \geq 0 \) on \([a,b] \). Similarly, we have \( y_2 - y_1 \geq 0 \), and thus \( y_1 = y_2 \). Then the proof is completed.
Now, we are in a position to prove the validity of the monotone method for (1.1). A function \( \alpha \in E \) is said to be a lower solution of (1.1) if it satisfies

\[
\begin{align*}
\alpha''(t) &\geq f(t, \alpha(t), \alpha(w(t))), & t \in I, \\
\alpha(0) &= \alpha(T), & \alpha'(0) \geq \alpha'(T).
\end{align*}
\]

An upper solution of (1.1) is defined analogously by reversing the inequalities above.

**Theorem 3.2.** Suppose that there exists a lower solution \( \alpha \) and an upper solution \( \beta \) of (1.1) such that \( \beta \leq \alpha \) on \([a, b] \). Assume that there exist two constants \( M \in (0, \left(\frac{\pi}{2}\right)^2), N > 0 \) satisfying

- \((H_1)\) \( f(t, u_2, v_2) - f(t, u_1, v_1) \geq -M(u_2 - u_1) - N(v_2 - v_1), \) for \( t \in I, \) whenever \( \beta(t) \leq u_1 \leq u_2 \leq \alpha(t) \) and \( \beta(w(t)) \leq v_1 \leq v_2 \leq \alpha(w(t)); \)
- \((H_2)\) \( \frac{N}{M} (\text{sec} \sqrt{\frac{N}{M}} - 1) < 1. \)

Then there exist two sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \), nonincreasing and nondecreasing respectively, with \( \alpha_0 = \alpha \) and \( \beta_0 = \beta \), which converge uniformly and monotonically to the extremal solution of (1.1) in the segment \([\beta, \alpha]\).

**Proof.** For each given \( \eta \in [\beta, \alpha] \), we consider the boundary value problem (3.1) with

\[ \sigma(t) = \sigma_{\eta}(t) = f(t, \eta(t), \eta(w(t))) + M\eta(t) + N\eta(w(t)). \]

We now refer this problem as \((PL)_{\eta}\).

Since \( \eta \in [\beta, \alpha] \), we have by \((H_1)\) and the definitions of lower and upper solutions that

\[
\alpha''(t) + M\alpha(t) + N\alpha(w(t)) \geq f(t, \alpha(t), \alpha(w(t))) + M\alpha(t) + N\alpha(w(t)) \geq f(t, \eta(t), \eta(w(t))) + M\eta(t) + N\eta(w(t)) = \sigma_{\eta}(t)
\]

and

\[
\beta''(t) + M\beta(t) + N\beta(w(t)) \leq \sigma_{\eta}(t).
\]

As a consequence, \( \alpha \) and \( \beta \) are respectively a lower and an upper solution for \((PL)_{\eta}\). Applying Theorem 3.1 permits us to define the operator \( A : [\beta, \alpha] \to [\beta, \alpha] \), where \( A\eta \) is the unique solution of \((PL)_{\eta}\) on \([\beta, \alpha]\).

Applying Theorem 3.1, it is easy to prove the following property of the operator \( A \):

- \((P_3)\) \( A \) is a monotone increasing mapping in the segment \([\beta, \alpha]\), namely, \( A\eta_1 \leq A\eta_2 \) when \( \eta_1, \eta_2 \in [\beta, \alpha] \) and \( \eta_1 \leq \eta_2 \).

Thus, we may define the sequences \( \{\alpha_n\}, \{\beta_n\} \) by \( \alpha_{n+1} = A\alpha_n \), \( \beta_{n+1} = A\beta_n \), \( \alpha_0 = \alpha, \beta_0 = \beta \). Using \((P_3)\), it is immediate to verify that

\[
\beta = \beta_0 \leq \beta_1 \leq \ldots \leq \beta_n \leq \ldots \leq \alpha_n \leq \ldots \leq \alpha_0 = \alpha,
\]

Since \( \{\alpha_n\} \) is nonincreasing, \( \{\beta_n\} \) is nondecreasing, \( \{\alpha_n'\} \) and \( \{\beta_n'\} \) is bounded in \( C(I, R) \), we have

\[
\lim_{n \to \infty} \alpha_n(t) := \alpha^*(t) \quad \text{and} \quad \lim_{n \to \infty} \beta_n(t) := \beta^*(t)
\]
uniformly and monotonically on \([a, b]\). Using the definition of \((PL_n)\) and passing the limit when \(n\) tends to \(\infty\), we conclude that \(\alpha^*(t)\) and \(\beta^*(t)\) are both solutions to the problem (1.1).

Furthermore, if \(y \in [\beta, \alpha]\) is a solution to the problem (1.1), then, by induction, \(\beta_n(t) \leq y(t) \leq \alpha_n(t)\) on \([a, b]\), \(n = 0, 1, 2, \cdots\), and hence, \(y \in [\beta^*, \alpha^*]\). This shows that \(\alpha^*(t)\) and \(\beta^*(t)\) are respectively maximal and minimal solutions to the problem (1.1) in the segment \([\beta, \alpha]\). \(
\)

Next we are in a position to prove the validity of the monotone method for (1.2). A function \(\alpha \in X\) (\(X\) is as in section 2) is said to be a lower solution to (1.2), if it satisfies

\[\alpha''(t) \geq f(t, \alpha(t), \alpha(w(t))), \quad t \in R.\]  (3.4)

An upper solution for (1.2) is defined analogously by reversing the inequalities above.

By using a same argument, we obtain the following result:

**Theorem 3.3.** Suppose that there exists a lower solution \(\alpha\) and an upper solution \(\beta\) of (1.2) such that \(\beta \leq \alpha\) in \(R\). Assume that there exist two constants \(M \in (0, \frac{\pi}{T})^2\), \(N > 0\) satisfying

\[(h_1) \quad f(t, u_2, v_2) - f(t, u_1, v_1) \geq -M(u_2 - u_1) - N(v_2 - v_1), \quad t \in R,\]

whenever \(\beta(t) \leq u_1 \leq u_2 \leq \alpha(t)\) and \(\beta(w(t)) \leq v_1 \leq v_2 \leq \alpha(w(t));\)

\[(h_1) \quad \frac{N}{M} (\sec \frac{T\sqrt{M}}{2} - 1) < 1.\]

Then there exist two sequences \(\{\alpha_n\}\) and \(\{\beta_n\}\), nonincreasing and nondecreasing, respectively, with \(\alpha_0 = \alpha\) and \(\beta_0 = \beta\), which converge uniformly and monotonically to the extremal \(T\)-periodic solution to (1.2) in the segment \([\beta, \alpha]\).

**References**


