Asymptotic Stability for a Class of Integro-differential Equations with Infinite Delay

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Abstract: Using the stability criteria in a Banach space BC rather than in an admissible space, we discuss the asymptotic behaviors for a class of integro-differential equations with infinite delay which are models of some population dynamics arising from hematology or ecology.

Keywords: Asymptotic stability; Integro-differential equation; Admissible space

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1. Introduction

In this paper, we investigate the integro-differential equation
\[ \frac{dN(t)}{dt} = -\gamma N(t) + \int_0^\infty K(s) F(N(t-s)) \, ds, \quad t \geq 0, \] (1.1)
in which \( F(u) \) is one of the following four forms:
\[ F(u) = \frac{1}{1 + u^n}, \quad n \geq 1; \] (1.2)
\[ F(u) = \frac{u}{1 + u^n}, \quad n \geq 1; \] (1.3)
\[ F(u) = e^{-\beta u}, \quad \beta > 0; \] (1.4)
\[ F(u) = ue^{-\beta u}, \quad \beta > 0; \] (1.5)
where \( \gamma > 0, \alpha > 0, n \geq 1, \beta > 0 \) are parameters; \( K: [0, \infty) \to [0, \infty) \) is assumed to be integrable and normalized such that
\[ \int_0^\infty K(s) \, ds = 1. \] (1.6)

Equation (1.1) is a generalization of the dynamical models with finite delays. We mention here that the background and derivation of model (1.1) with (1.2)-(1.5) can be found in [6, 8-11] and some investigations about oscillations and global attractivity of models with finite delays can...
be found in \([4, 6, 8-11]\). Let \( C = C \left[ \left( - \infty, 0 \right], \mathbb{R} \right] \) and
\[
BC = \left( C, \left\| \cdot \right\| \right) = \left\{ \psi \in C : \left\| \psi \right\| = \sup_{- \infty < s \leq 0} \left| \psi(s) \right| < \infty \right\},
\]
\[
G = \left\{ g \in C \left( - \infty, 0 \right], [1, 1 \infty) : g(0) = 1, \lim_{s \to 0^-} g(s) = + \infty, g(s) \text{ is nonincreasing} \right\},
\]
\[
C_\varepsilon = \left( C_\varepsilon, \left\| \cdot \right\|_\varepsilon \right) = \left\{ \psi \in C : \left\| \psi \right\|_\varepsilon = \sup_{- \infty < s \leq 0} \left| \frac{\psi(s)}{g(s)} \right| < \infty \right\}, \text{ where } g \in G,
\]
\[
B(r) = \left\{ \psi \in BC : \left\| \psi \right\|_r < r \right\}, \quad \Box B_\varepsilon(r) = \left\{ \psi \in C_\varepsilon : \left\| \psi \right\|_\varepsilon < r \right\},
\]
\[
UC_\varepsilon = \left\{ \psi \in C_\varepsilon : \left| \frac{\psi}{g} \right| \text{ is uniformly continuous on } \left( - \infty, 0 \right]\right\}, \text{ where } g \in G \text{ and } \lim_{s \to 0^-} g(s) = 1 \text{ uniformly on } \left( - \infty, 0 \right]\}
\]

It is known \(^2\) that both \( BC \) and \( C_\varepsilon \) are Banach spaces. In order to discuss a functional differential equation (abbrev. as FDE) with infinite delay, one needs an admissible space \((13) \) & \((7)\) as the phase space to establish the standard existence-uniqueness-continuous dependence-continuation theorems. There are a variety of choices of admissible space for FDE with infinite delay. One of them is the subspace \( UC_\varepsilon \) of \( C_\varepsilon \) which was introduced by Burton and others for Volterra equations \(^1\). Gopalsamy and Weng \(^5\) discussed the global attractivity and oscillations of \((1.1)\) with the form \((1.2)\) in \( BC \). The problem is that the authors did not mind if \((1.1)-(1.2)\) has the basic properties such as local existence, uniqueness etc. However, for most systems arising from applied science like \((1.1)\), \( BC \) seems to be a natural phase space and more convenient for use. But \( BC \) is not an admissible space and the positive orbit of a bounded solution is not always precompact in it. Feng \(^3\) used the concept of "fading memory" and "integral fading memory" to discuss the autonomous FDE
\[
x(t) = f(x_t) \tag{1.7}
\]
with infinite delay so that the basic properties of which can locally extend from \( BC \) to \( C_\varepsilon \), and then gave criteria which can be examined only in \( BC \) for globally asymptotic behavior of \((1.7)\). In this article, we shall show that \((1.1)\) satisfies the conditions in these criteria under some choice of parameters and \( K(s) \). Thus the equilibria of \((1.1)\) are globally asymptotically stable.

It could be understood that the solution of \((1.1)\) with the positive initial condition remains positive as long as it exists. Thus we only consider \( F(u) \) for \( u \geq 0 \) in what follows.

## 2 Functional with Fading Memory

**Definition 2** \(^1\) \() \quad \text{Functional} f : \left[ 0, \infty \right) \times BC \rightarrow \mathbb{R} \text{ is said to have a fading memory locally relative to } g \text{(abbrev. as FMLR-g)} \text{ if for every } r > 0, \text{ there is a } g \in G \text{ such that } f \text{ is defined in } \left[ 0, \infty \right) \times B_\varepsilon(r) \text{ and for } \varepsilon > 0, T > 0, \text{ there exists } K > 0 \text{ such that } \left\{ \psi, \varphi \in B_\varepsilon(r), \psi(s) = \varphi(s) \right\} \text{ on } [-K, 0] \text{ implies that } \left| f(t, \psi) - f(t, \varphi) \right| < \varepsilon \text{ for all } t \in [0, T] \right\}

**Definition 2** \(^2\) \() \quad Q : \left[ 0, \infty \right) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \text{ is said to have integral fading memory if for each } \varepsilon > 0, \text{ and each } r > 0 \text{ there exists } L > 0 \text{ such that} \int_{s}^{K} Q(t, t + s, \psi(s)) \, ds < \varepsilon \text{ for all } t \in [0, T] \right\}
for all $t \geq 0$ and all $\Psi \in BC$ with $\| \Psi \| \leq r$.

**Lemma 2** If $Q$ is continuous and has integral fading memory, then for every $r > 0$, there exists a $g \in G$ such that

$$\int_{-\infty}^{\infty} |Q(t, t + s, \Psi(s))| ds \to 0 \text{ as } K \to +\infty$$

uniformly for $t \geq 0$ and $\Psi \in G$ with $\| \Psi \| \leq r$.

**Remark 1** We can see from Definition 2.1-2.2 and Lemma 2.1 that if $Q$ is continuous and has integral fading memory, then $F(t, \Psi) = \int_{-\infty}^{0} Q(t, t + s, \Psi(s)) ds$ has FMLR-$g$.

**Remark 2** If $g_1 \in G$ satisfies the requirements of Definition 2.1, then every $g \in G$ satisfying $g(t) \leq g_1(t)$ for all $t \in (-\infty, 0]$ can play the same role as $g_1(t)$.

Let $f : BC \to \mathbb{R}$ be defined by

$$f(\Psi) = -\Psi(0) + \int_{-\infty}^{0} K(-s) F(\Psi(s)) ds$$

Then (1.1) can be rewritten as

$$\frac{dY(t)}{dt} = f(N(t)), \quad t \geq 0, \quad (2.1)$$

where $N(\Theta) = N(t + \Theta), \quad -\infty < \Theta \leq 0$. The positive equilibrium $N^*$ of (2.1) is the positive solution of the equation:

$$\gamma_1 = \alpha F(\eta). \quad (2.2)$$

If $N^*$ is the positive equilibrium of (1.1). Let $y = N - N^*$. Then (1.1) is changed into

$$\frac{dy(t)}{dt} = H(y), \quad t \geq 0, \quad (2.3)$$

where $H : BC \to \mathbb{R}$ is defined by

$$H(\Psi) = -\Psi(0) + \int_{-\infty}^{0} K(-s) [F(N^* + \Psi(s)) - F(N^*)] ds$$

We claim that $H(\Psi)$ satisfies the following conditions:

(A.1) $H(0) = 0$;

(A.2) $H$ has FMLR-$g$;

(A.3) $H(\Psi)$ satisfies local Lipschitz condition in $BC$.

Thus we obtain from (A.1)-(A.3) that for each $r > 0$, there exists a $g \in G$ such that the existence-uniqueness-continuous dependence-continuation of solutions of (2.3) holds in $B_{g}(r)$.[1]

The same conclusion holds for equation (2.1).

### 3. Asymptotic Stability of Equilibrium

We give a lower bound estimate for solutions of (2.1) when

$$F(u) = \frac{u^{n+1}}{1 + u^n} (n \geq 1) \quad \text{or} \quad F(u) = ue^{-\beta u} (\beta > 0).$$

The idea for proving our conclusions for both cases are similar. We shall only give a little details for $F(u) = \frac{u^{n+1}}{1 + u^n}$ and omit the others. Consider a function
\[ g(u) = -\gamma + \frac{\alpha}{1 + u^n}, u \geq 0 \]  

Suppose that \( \alpha > \gamma \), then we have

\[ g(N^s) = 0, N^s = \left( \frac{\alpha}{\gamma} - 1 \right)^{1/n} \]  

and \( g(u) > 0 \) for \( u \in (0, N^s) \).

In view of the property of \( F(u) \), one can choose a constant \( 0 < b < 1 \) such that

\[ \begin{cases} 
    \text{sup} \{N^0, u^s\} \text{ as } n > 1, \\
    N^0 \triangleq \ln u^s < \frac{\alpha}{\gamma}N^0 \text{ as } n = 1,
\end{cases} \]

and

\[ F(N_0) \leq F(N^0), F(u) \geq F(N_0) \text{ for } u \in [N_0, N^0], \]

where\( N^0 = \frac{\alpha}{\gamma}, u^s = \left( \frac{1}{n-1} \right)^{1/n} \).

**Remark 3**  

If \( F(u) = u e^{\beta u} (\beta > 0) \), (3.1) and (3.3) are changed into

\[ g(u) = -\gamma + \alpha \exp(-\beta u), u \geq 0, \]

\[ \begin{cases} 
    \text{sup} \{N^0, u^s\}, \text{ where } u^s = \frac{1}{\beta}N^0, N^s = \frac{1}{\beta} \ln \frac{\alpha}{\gamma}, N^0 = \frac{\alpha}{\beta e} \end{cases} \]

**Lemma 3**  

Assume that \( N(t) \) is any positive solution of (2.1) and \( \frac{\alpha}{\gamma} > 1 \), then there exists a \( T > 0 \) such that

\[ N(t) \geq N_0 \text{ for } t \geq T, \]

where \( N_0 = \ln u^s \) is defined to satisfy (3.3)-(3.4) or (3.4)-(3.5) according to the different \( F(u) \) in (1.3) or (1.5).

Let

\[ BC = \{ \Psi \in BC, \Psi(s) \geq 0 \text{ for } s \in (-\infty, 0] \}, \]

\[ BC_* = \{ \Psi \in BC, \exists s_0 \in (-\infty, 0] \text{ such that } \Psi(s_0) \neq 0 \}. \]

We shall investigate the asymptotic stability of equilibria of (2.1). We need to consider the following two cases:

1. (2.1) has zero equilibrium \( N(t) \equiv 0 \);
2. (2.1) has a positive equilibrium \( N(t) \equiv N^s \).

First, we note that case 1 arises while

\[ F(u) = \frac{u}{1 + u^n}(n \geq 1) \text{ or } F(u) = u e^{\beta u} (\beta > 0). \]

**Theorem 3**  

Assume that (3.7) holds and \( \gamma > \alpha \)

\[ \int_0^\infty sK(s)ds < \infty. \]

Then the zero solution of (2.1) is globally asymptotically stable in \( BC_* \). The attraction region is \( BC_* \).

**Proof**  

Let functional \( V(\Psi) \) be defined by

\[ V(\Psi) = \Psi(0) + \int_{-\infty}^0 \left( \int_0^s K(u)du \right) F(\Psi(s))ds \text{ for } \Psi \in BC_* \]

Note that when \( \Psi(s) = N(s) \), we have

\[ \left| \int_0^\infty \left( \int_0^s K(u)du \right) F(\Psi(s))ds \right| \leq \max_{s \geq 0} |F(u)| \int_0^\infty sK(s)ds, \]
which together with the convergence condition (3.8) implies that \( V(\Psi) \) has \( FM LR - g \). On the other hand, we have

\[
u(r) \triangleq r \to + \infty \text{ as } r \to + \infty \text{ and } u(\|\Psi(0)\|) \leq V(\Psi).
\]

(3.9)

Now let's calculate the rate of change of \( V \) along the solution \( N(t) \) of (2.1), and we obtain

\[
\frac{dV(N_r)}{dt} = -\mathcal{Y}(N_r) + \alpha F(N_r).
\]

(3.10)

We obtain from (3.10), (3.7) and the fact \( N(t) \geq 0 \) that \( \frac{dV(N_r)}{dt} \leq -\mathcal{Y}(N_r) \). Define \( \psi(r) = (\mathcal{Y}-\alpha)N(t) \). Define \( u(r) = \psi(r) \). Then we get from (3.9) that

\[
u(\|\Psi(0)\|) \leq V(\Psi) \leq -w(\|\Psi(0)\|) \text{ for all } \Psi \in BC_+.
\]

We conclude from Theorem 4.2 of [3] that the zero solution of (2.1) is globally asymptotically stable, and \( BC_+ \) is its attraction region. Thus we complete the proof.

Secondly, we investigate the case when (2.1) has a positive equilibrium \( N(t) \equiv N^* \). This arises from the following cases:

\[
F(u) = \frac{1}{1 + u^n(n \geq 1)}; \quad F(u) = \frac{u}{1 + u^n(n \geq 1)}; \quad F(u) = \alpha F(N(t)), \quad F(u) = \alpha F(N(t)).
\]

We consider equation (2.3) as follows:

\[
\frac{dV(y(t))}{dt} = -\mathcal{Y}(y(t)) + \int_0^\infty K(s) [F(N^* + y(t-s)) - N^*]ds, t \geq 0
\]

(3.11)

Then \( y(t) \equiv 0 \) is the solution of (3.11). Let

\[
V(\Psi) = \|\Psi(0)\| + \int_0^\alpha \int_0^\infty K(u)du |F_1(\Psi(s))|ds \text{ for } \Psi + N^* \in BC_+.
\]

where \( F_1(\Psi(s)) = F(N^* + \Psi(s)) - F(N^*) \). Then along the solution of (3.11), we have

\[
\frac{dV(y(t))}{dt} \leq -\mathcal{Y}[y(t)] + \alpha |F_1(y(t))|
\]

(3.12)

We only discuss the situation when \( F(u) = \frac{1}{1 + u^n(n \geq 1)} \), then

\[
|F_1(y(t))| = \frac{n\xi^{n-1}(t)}{[1 + \xi^n(t)]^2} |y(t)|,
\]

(3.13)

where \( \xi(t) \geq 0 \) lies between \( N^* \) and \( N^* + y(t) \). Note the fact:

\[
\frac{x^{n-1}}{(1 + x^n)^2} \leq \begin{cases} x^{n-1} & \text{for } 0 \leq x \leq 1, \\ \frac{x^n}{(1 + x^n)^2} & \text{for } x > 1, \end{cases}
\]

which together with (3.12)-(3.13) leads to

\[
\frac{dV(y(t))}{dt} \leq -\mathcal{Y}N^* |y(t)|
\]

(3.14)

If \( \mathcal{Y}N^* \), then \( N = N^* \) of (2.1) is globally asymptotically stable in \( BC_+ \).

**Theorem 3.2** Any one of the following is the sufficient condition for the equilibrium \( N = N^* \) of (2.1) to be globally asymptotically stable in \( BC_+ \). The attraction region is \( BC_+ \).

1. \( \mathcal{Y}N^* \), \( F(u) = \frac{1}{1 + u^n(n \geq 1)} \);
2. $\gamma > \alpha \beta, F(u) = e^{\beta u} (\beta > 0);$

3. $\gamma < \alpha < \frac{\gamma}{\alpha}, F(u) = \frac{u}{A} (n \geq 1), \text{ where } A = \max \left\{ \left( \frac{n - 1}{4n} \right)^2, h(N_0) \right\}, h(u) = \frac{1}{(1 + u^n)^{1/n}}, N_0 = b \left( \frac{\gamma}{\alpha} - 1 \right), b \in (0, 1) \text{ is some constant satisfying (3.3)}$;

4. $\gamma < \alpha < \frac{\gamma}{\alpha}, F(u) = u e^{\beta u} (\beta > 0), \text{ where } A = \max \{ e^2, 1 - \beta N_0 \}, N_0 = \frac{b}{\beta} \ln \frac{\alpha}{\gamma}, b \in (0, 1) \text{ is some constant satisfying (3.5)}.$

References:


