Chapter 2. Parabolic obstacle problem  
— American put option

The problem of American put option is to find \( V(s, \tau) \) satisfying

\[
\begin{cases}
\partial_\tau V - \frac{\sigma^2}{2} s^2 \partial_{ss} V - (r - q)s \partial_s V + rV \geq 0, & V - (k - s)^+ \geq 0, \quad s > 0, \quad 0 < \tau < T \\
\left[ \partial_\tau V - \frac{\sigma^2}{2} s^2 \partial_{ss} V - (r - q)s \partial_s V + rV \right] \left[ (V - (k - s)^+) \right] = 0, \\
V(s, 0) = (k - s)^+ \quad s \geq 0.
\end{cases}
\]  

(0.1)

where \( V \) is the market price of American put option, \( s \) is the price of risk asset(stock), \( \tau = T - t \), where \( t \) is the time and \( T \) is expiry of the option. \( \sigma, r, q, k \) are positive constants representing volatility, interest rate, dividend and strike price respectively. According to Fichera’s theorem, we must not put the boundary condition on the boundary \( s = 0 \).

If we set \( x = \ln s, \ u(x, \tau) = V(s, \tau) \), then

\[ s\partial_s V = \partial_x u, \quad s^2 \partial_{ss} V = \partial_{xx} u - \partial_x u \]

thus the problem (0.1) becomes

\[
\begin{cases}
\partial_\tau u - \frac{\sigma^2}{2} \partial_{xx} u - (r - q - \frac{\sigma^2}{2}) \partial_x u + ru \geq 0, & u - (k - e^x)^+ \geq 0, \quad x \in \mathbb{R}, \quad 0 < \tau < T \\
\left[ \partial_\tau u - \frac{\sigma^2}{2} \partial_{xx} u - (r - q - \frac{\sigma^2}{2}) \partial_x u + ru \right] \left[ (u - (k - e^x)^+) \right] = 0, \\
u(x, 0) = (k - e^x)^+, \quad x \in \mathbb{R}.
\end{cases}
\]  

(0.2)

1. The existence and uniqueness of \( W_{p,loc}^{2,1} \) solution

In order to prove the existence of solution of (0.2), construct a penalty function \( \beta_\varepsilon(t) \) (see figure 1) satisfying

\[
\begin{align*}
\beta_\varepsilon(t) & \in C^2(-\infty, +\infty), \quad \beta_\varepsilon(t) \leq 0, \\
\beta_\varepsilon(0) & = -C_0 (C_0 > 0, \text{ for its definition, see (1.7)}) \\
\beta'_\varepsilon(t) & \geq 0, \quad \beta''_\varepsilon \leq 0,
\end{align*}
\]

and

\[
\lim_{\varepsilon \to 0} \beta_\varepsilon(t) = \begin{cases} 
0, & t > 0 \\
-\infty, & t < 0
\end{cases}
\]

figure 1
Since the smoothness of \((k - e^x)^+\) is not good enough, so we need to define a function 
\(\pi_\varepsilon(t)\) (see figure 2) 
\[
\pi_\varepsilon(t) = \begin{cases} 
t & t \geq \varepsilon \\
\varepsilon & |k| \leq \varepsilon \\
0 & t \leq -\varepsilon \end{cases}
\]
\(\pi_\varepsilon(t) \in C^\infty,\ 0 \leq \pi'_\varepsilon \leq 1, \ \pi''_\varepsilon(t) \geq 0, \ \lim_{\varepsilon \to 0} \pi_\varepsilon(t) = t^+.
\)

Consider an approximation problem of (0.2) 
\[
\pi_\varepsilon \times (0, T).
\]

\(\Omega_\varepsilon = IR \times (0, T)\). Since \(\Omega_\varepsilon\) is an unbounded domain, we apply a bounded domain to approximate \(\Omega_\varepsilon\): 
\[
\begin{cases} 
\sigma^2 u_x - (r - q - \sigma^2)u_x + ru + \beta_\varepsilon(u - \pi_\varepsilon(k - e^x)) = 0, & x \in IR, \ 0 < \tau < T \\
u(x, 0) = \pi_\varepsilon(k - e^x), & x \in IR.
\end{cases}
\]

Denote \(\Omega_T = IR \times (0, T)\). Since \(\Omega_T\) is an unbounded domain, we apply a bounded domain to approximate \(\Omega_T\): 
\[
\begin{cases} 
\sigma^2 u_x - (r - q - \sigma^2)u_x + ru + \beta_\varepsilon(u - \pi_\varepsilon(k - e^x)) = 0, & (x, \tau) \in \Omega^R_T \\
u(x, \tau) = \pi_\varepsilon(k - e^x), & \tau \in \partial_\varepsilon \Omega^R_T
\end{cases}
\]

where \(\Omega^R_T = (-R, R) \times (0, T)\), \(\partial_\varepsilon \Omega^R_T\) is the parabolic boundary of \(\Omega^R_T\).

**Lemma 1.1:** For fixed \(\varepsilon, R\), problem (1.2) has a unique solution \(u = u_{\varepsilon,R} \in W^{2,1}_p(\Omega^R_T), 1 < p < +\infty\).

**Proof:** We apply Schauder fixed point theorem to prove the existence of nonlinear problem (1.2). Denote \(B = C(\bar{\Omega}^R_T), \ D = \{v \in B| v \geq 0\}\), then \(D\) is a closed convex set in \(B\). 
\(\forall v \in D, \) define a mapping \(u = Fv\) is the solution of following linear problem 
\[
\begin{cases} 
\mathcal{L}u = -\beta_\varepsilon(v - \pi_\varepsilon(k - e^x)) & \text{in } \Omega^R_T, \\
u = \pi_\varepsilon(k - e^x) & \text{on } \partial_\varepsilon \Omega^R_T.
\end{cases}
\]

where 
\[
\mathcal{L}u = u_x - \sigma^2 u_x - (r - q - \sigma^2)u_x + ru.
\]

The linear problem (1.3) has a unique solution \(u \in W^{2,1}_p(\Omega^R_T)\), moreover 
\[
|u|_{W^{2,1}_p(\Omega^R_T)} \leq C(|\beta_\varepsilon(v - \pi_\varepsilon(k - e^x))|_{L^p(\Omega^R_T)} + |\pi_\varepsilon(k - e^x)|_{W^{2,1}_p(\Omega^R_T)}),
\]

where \(C\) is independent of \(\varepsilon\).

In the following we prove \(F\) satisfying 
(1). \(F(D) \subset D\),  
(2). \(F(D)\) is compact set in \(B\),  
(3). \(F\) is continuous.

In fact 
(1). It follows that \(\mathcal{L}u \geq 0\) by \(\beta_\varepsilon(v - \pi_\varepsilon(k - e^x)) \leq 0\), hence \(u \geq 0\), \(i.e.,\ \ F(D) \subset D\). 
(2). From \(v \geq 0\) we know that \(\beta_\varepsilon\) is bounded, therefore \(\exists C_\varepsilon > 0, \) such that 
\[
|u|_{W^{2,1}_p(\Omega^R_T)} \leq C_\varepsilon
\]
Embedding theorem reduced that \( F(D) \) is precompact in \( B \) (if \( p \) is big enough).

(3). According to the definition, we should prove if
\[
D \ni v_j \longrightarrow v \quad \text{in} \quad D \\
F(v_j) = u_j, \quad F(v) = u,
\]
then
\[
\lim_{j \to \infty} u_j = u \quad \text{in} \quad D
\]
In fact \( u_j - u \) is the solution of
\[
\begin{cases}
L(u_j - u) = -\beta'(\bullet)(v_j - v) & \text{in} \quad \Omega_T^R \\
u_j - u = 0 & \text{on} \quad \partial_p \Omega_T^R
\end{cases}
\]
therefore, by the estimate of maximum norm,
\[
|u_j - u|_{L^\infty(\Omega_T^R)} \leq C|v_j - v|_{L^\infty(\Omega_T^R)},
\]
Applying Schauder fixed point theorem we know that \( \exists u \in W^{2,1}_p(\Omega_T^R) \) satisfying (1.2).

We left the proof of uniqueness as exercise.

**Lemma 1.2.** For the solution \( u = u_{e,R} \) of (1.2), the following uniform estimates hold
\[
\begin{align*}
\pi_e(k - e^x) &\leq u_{e,R} \leq M_0, \quad (1.5) \\
-C_0 &\leq \beta_e(u_{e,R} - \pi_e(k - e^x)) \leq 0, \quad (1.6)
\end{align*}
\]
where \( C_0, M_0 \) are independent of \( e, R \)

**Proof:** Since
\[
\begin{align*}
\frac{d}{dx}\pi_e(k - e^x) &= \pi'_e(k - e^x)(-e^x) \\
\frac{d^2}{dx^2}\pi_e(k - e^x) &= \pi''_e(k - e^x)(-e^x)^2 + \pi'_e(k - e^x)(-e^x)
\end{align*}
\]
so
\[
\begin{align*}
L\pi_e(k - e^x) &= -\frac{\sigma^2}{2}\pi''_e(k - e^x)e^{2x} + (r - q)\pi'_e(k - e^x)e^x + r\pi_e(k - e^x) \\
&\leq (r - q)\pi'_e(k - e^x)e^x + r\pi_e(k - e^x)
\end{align*}
\]
it is bounded from above by a fact that
\[
\pi_e(k - e^x) \begin{cases}
\leq e, & \text{if} \quad k - e^x \leq e \\
=k - e^x \leq k, & \text{if} \quad k - e^x > e
\end{cases}
\]
so
\[
0 \leq \pi_e(k - e^x) \leq k + \varepsilon
\]
and 
\[
\pi'_\varepsilon(k - e^x)e^x \begin{cases} 
= 0, & \text{if } k - e^x \leq -\varepsilon \\
\leq e^x \leq k + \varepsilon, & \text{if } k - e^x > -\varepsilon 
\end{cases}
\]

thus 
\[
0 \leq \pi'_\varepsilon(k - e^x)e^x \leq k + \varepsilon
\]

therefore 
\[
(r - q)\pi'_\varepsilon(k - e^x)e^x + r\pi_\varepsilon(k - e^x) \leq |r - q|(k + \varepsilon) + r(k + \varepsilon)
\]

so there exists \( C_0 > 0 \), which is independent of \( \varepsilon \), such that 
\[
L\pi_\varepsilon(k - e^x) \leq C_0, \quad (1.7)
\]

It is reduced that \( \pi_\varepsilon(k - e^x) \) is a subsolution of the problem (1.2), thus we proved the left part of the inequality (1.5). From this and the definition of \( \beta_\varepsilon \) we know that (1.6) is true.

Backing to the problem (1.2), since \( \beta_\varepsilon \) is bounded, so \( \exists M_0 > 0 \), such that 
\[
u_\varepsilon,R \leq M_0
\]

**Theorem 1.3.** For fixed \( \varepsilon > 0 \), problem (1.1) has a unique solution \( u = u_\varepsilon \in C(\overline{\Omega}_T) \), and \( \forall R > 0, u_\varepsilon \in W^{2,1}_p(\Omega^R_T) \).

**Proof:** We prove that when \( R \to +\infty \), the solutions of (1.2):
\[
\begin{aligned}
&\mathcal{L}u_\varepsilon,R + \beta_\varepsilon(u_\varepsilon,R - \pi_\varepsilon(k - e^x)) = 0 \quad \text{in } \Omega^R_T, \\
u_\varepsilon,R = \pi_\varepsilon(k - e^x) \quad \text{on } \partial_p\Omega^R_T,
\end{aligned}
\]

converge to the solution of (1.1).

Letting \( R = 1, 2, \ldots \), we have that on \( \Omega^1_T \), by \( W^{2,1}_p \) estimate,
\[
|u_{\varepsilon,j}|_{W^{2,1}_p(\Omega^1_T)} \leq C, \quad j = 1, 2, \ldots
\]

where \( C \) is independent of \( j \), therefore there is a subsequence \( \{u_{\varepsilon,j}^{(1)}\} \) of \( \{u_{\varepsilon,j}\} \), such that 
\[
u_{\varepsilon,1}^{(1)}, \nu_{\varepsilon,2}^{(1)}, \ldots \nu_{\varepsilon,k}^{(1)} \rightarrow u_\varepsilon^{(1)} \in W^{2,1}_p(\Omega^1_T) \quad \text{(in } W^{2,1}_p(\Omega^1_T) \text{ weakly, and in } C(\overline{\Omega}^1_T) \text{ uniformly)}
\]

On \( \Omega^2_T \) again by \( W^{2,1}_p \) estimate,
\[
|u_{\varepsilon,j}^{(1)}|_{W^{2,1}_p(\Omega^2_T)} \leq C
\]

\( C \) is independent of \( j \), therefore there exists a subsequence \( \{u_{\varepsilon,j}^{(2)}\} \) of \( \{u_{\varepsilon,j}^{(1)}\} \), such that 
\[
u_{\varepsilon,1}^{(2)}, \nu_{\varepsilon,2}^{(2)}, \ldots \nu_{\varepsilon,k}^{(2)} \rightarrow u_\varepsilon^{(2)} \in W^{2,1}_p(\Omega^2_T) \quad \text{(in } W^{2,1}_p(\Omega^2_T) \text{ weakly, and in } C(\overline{\Omega}^2_T) \text{ uniformly)},
\]

moreover 
\[
u_\varepsilon^{(2)} = u_\varepsilon^{(1)} \quad \text{in } \Omega^1_T
\]

\[\ldots\ldots\]
On \( \Omega_T^m \) by \( W^{2,1}_p \) estimate,
\[
|u_{\varepsilon,j}^{(m-1)}|_{W^{2,1}_p(\Omega_T^m)} \leq C
\]

C is independent of j, therefore there exists subsequence
\[
u_{\varepsilon,1}^{(m)}, \nu_{\varepsilon,2}^{(m)}, \ldots, \nu_{\varepsilon,k}^{(m)}, \ldots \to \nu_{\varepsilon}^{(m)} \in W^{2,1}_p(\Omega_T^m) \text{ (in } W^{2,1}_p(\Omega_T^m), \text{ and in } C(\overline{\Omega_T^m}) \text{ uniformly),}
\]
moreover
\[
u_{\varepsilon}^{(m)} = u_{\varepsilon}^{(j)} \quad \text{in } \Omega_T^j \quad 1 \leq j \leq m - 1
\]
\[
\ldots \ldots
\]

Define \( u_{\varepsilon}(x, \tau) = u_{\varepsilon,1}^{(m)}(x, \tau) \), if \((x, \tau) \in \Omega_T^m\), then \( u_{\varepsilon}(x, \tau) \) is defined on \( \Omega_T \). Consider sequence \( u_{\varepsilon, m}^{(m)} \) in diagram, for \( \forall R > 0 \), if \( m \to \infty \),
\[
u_{\varepsilon, m}^{(m)} \to \nu_{\varepsilon} \in W^{2,1}_p(\Omega_T^R) \quad \text{(in } W^{2,1}_p(\Omega_T^R) \text{ weakly, and in } C(\overline{\Omega_T^R}) \text{ uniformly).}
\]

Let \( m \to \infty \) in the system
\[
\begin{cases}
L u_{\varepsilon, m}^{(m)} + \beta_{\varepsilon}(u_{\varepsilon, m}^{(m)} - \pi_{\varepsilon}(k - e^x)) = 0 & \text{in } \Omega_T^R \\
u_{\varepsilon, m}^{(m)} = \pi_{\varepsilon}(k - e^x) & \text{on } \partial_p \Omega_T^R
\end{cases}
\]
we find that \( u_{\varepsilon}(x, \tau) \) is the solution of (1.1).

The proof of uniqueness is the same as in lemma 1.1.

**Theorem 1.4.** The problem (0.2) has a unique solution \( u \in C(\overline{\Omega_T}), \forall R > 0, \forall \delta > 0, \) \( u \in W^{2,1}_p(\Omega_T^R \setminus B_\delta(P_0)) \), moreover \( u_{\varepsilon} \) uniformly converge to \( u \) on \( \Omega_T^R \). Where \( P_0 = (\ln k, 0), B_\delta(P_0) \) is a disk with center \( P_0 \) and radius \( \delta \).

**Proof:** Since the constants in the estimates (1.5), (1.6) are independent of \( \varepsilon, R \), so if \( R \to \infty \), then
\[
\pi_{\varepsilon}(k - e^x) \leq u_{\varepsilon} \leq M_0 \quad \text{(1.8)}
\]
\[
-C_0 \leq \beta_{\varepsilon}(u_{\varepsilon} - \pi_{\varepsilon}(k - e^x)) \leq 0 \quad \text{(1.9)}
\]
where \( C_0, M_0 \) are independent of \( \varepsilon \).

Applying \( C^{\alpha, \alpha/2} \) estimate, we obtain
\[
|u_{\varepsilon}|_{C^{\alpha, \alpha/2}(\overline{\Omega_T^R})} \leq C \quad \text{C is independent of } \varepsilon.
\]

hence \( \exists u \in C(\overline{\Omega_T}), \) such that \( u_{\varepsilon} \to u \) \( \text{in } C(\overline{\Omega_T^R}) \). From (1.9) we see that \( \forall R > 0, \forall \delta > 0 \)
\[
|u_{\varepsilon}|_{W^{2,1}_p(\Omega_T^R \setminus B_\delta(P_0))} \leq C
\]
where \( C \) is independent of \( \varepsilon \) and depends on \( R, \delta \). Applying the same method (abstracting diagram subsequence), we know that there exists a subsequence of \( \{u_{\varepsilon}\} \) (still denoted by \( \{u_{\varepsilon}\} \)) and \( u \in W^{2,1}_p(\Omega_T^R \setminus B_\delta(P_0)) \), such that
\[
u_{\varepsilon} \to u \quad \text{in } W^{2,1}_p(\Omega_T^R \setminus B_\delta(P_0)) \text{ weakly},
\]
let $\varepsilon \to 0$ in $\mathcal{L}u_\varepsilon \geq 0$, then
\[
\mathcal{L}u \geq 0 \quad \text{in } \Omega_T \setminus B_\delta(p_0).
\]
notice that $R, \delta$ are arbitrary, so
\[
\mathcal{L}u \geq 0 \quad \text{in } \Omega_T
\]
It is reduced, by letting $\varepsilon \to 0$ in (1.8), that
\[
(k - e^x)^+ \leq u \leq M_0.
\]

At last we prove
\[
\mathcal{L}u = 0 \quad \text{in } \{u > (k - e^x)^+\},
\]
in fact, $\forall (x_0, \tau_0) \in \{u > (k - e^x)^+\}$,
\[
u(x_0, \tau_0) > (k - e^{x_0})^+,
\]
so $\exists \delta > 0$ and a neighborhood $U$ of $(x_0, \tau_0)$, if $\varepsilon$ is small enough,
\[
u_\varepsilon(x, \tau) > \pi_\varepsilon(k - e^x) + \delta, \quad (x, \tau) \in U,
\]
hence when $\varepsilon \to 0$,
\[
0 \geq \beta_\varepsilon(u_\varepsilon(x, \tau) - \pi_\varepsilon(k - e^x)) \geq \beta_\varepsilon(\delta) \to 0, \quad (x, \tau) \in U,
\]
therefore at $(x_0, \tau_0)$
\[
\mathcal{L}u = 0 \quad (x, \tau) \in U.
\]

The proof of uniqueness:
Suppose $u_1, u_2$ are two solutions of (0.2), and $\mathcal{N} = \{(x, \tau)|u_1(x, \tau) > u_2(x, \tau)\}$ is not an empty set, then on the set $\mathcal{N}$
\[
u_1 > u_2 \geq (k - e^x)^+
\]
so
\[
\mathcal{L}u_1 = 0 \quad \text{in } \mathcal{N}
\]
\[
\mathcal{L}u_2 \geq 0 \quad \text{in } \mathcal{N}
\]
hence
\[
\begin{cases}
\mathcal{L}(u_1 - u_2) \leq 0 \quad \text{in } \mathcal{N} \\
u_1 - u_2 = 0 \quad \text{on } \partial_p \mathcal{N}
\end{cases}
\]
it follows that $u_1 - u_2 \leq 0$ in $\mathcal{N}$ by maximum principle of subsolution. This is a contradiction to the definition of $\mathcal{N}$.
2. Properties of free boundary of American put option

From previous section we know that \( u(x, \tau) \) can be approximated by \( u_{\epsilon}(x, \tau) \) which is the solution of
\[
\begin{cases}
\partial_{\tau}u_{\epsilon} - \frac{\sigma^2}{2}\partial_{xx}u_{\epsilon} - (r - q - \frac{\sigma^2}{2})\partial_{x}u_{\epsilon} + ru_{\epsilon} + \beta_{\epsilon}(u_{\epsilon} - \pi_{\epsilon}(k - e^{\tau})) = 0, \\
u_{\epsilon}(x, 0) = \pi_{\epsilon}(k - e^{x}), \quad x \in \mathbb{R}.
\end{cases}
\tag{2.1}
\]

For fixed \( \epsilon > 0 \), applying a priorior estimates we have \( u_{\epsilon} \in C^{2+\alpha,1+\alpha/2}(\Omega_T) \), it means that \( \partial_{\tau}u_{\epsilon}, \partial_{xx}u_{\epsilon} \) and \( \partial_{x}u_{\epsilon} \) are bounded functions.

**Lemma 2.1.**
\[
\partial_{x}u(x, \tau) \leq 0, \quad (x, \tau) \in \Omega_T.
\tag{2.2}
\]

**Proof:** Differentiating equation (2.1) with respect to \( x \), denoting \( \partial_{x}u_{\epsilon} = v \), then
\[
\begin{cases}
\partial_{x}v - \frac{\sigma^2}{2}\partial_{xx}v - (r - q - \frac{\sigma^2}{2})\partial_{x}v + rv + \beta'_{\epsilon}(\cdot)v = \beta'_{\epsilon}(\cdot)\pi'_{\epsilon}(k - e^{x})(-e^{x}) \leq 0 \\
v(x, 0) = \pi'_{\epsilon}(k - e^{x})(-e^{x}) \leq 0, \quad x \in \mathbb{R}
\end{cases}
\tag{2.3}
\]

From maximum principle for subsolution we obtain \( \partial_{x}u_{\epsilon} = v \leq 0 \). (2.2) follows by \( \epsilon \to 0 \).

**Lemma 2.2.**
\[
\partial_{\tau}u(x, \tau) \geq 0, \quad (x, \tau) \in \Omega_T.
\tag{2.4}
\]

**Proof:** Differentiating (2.1) with respect to \( \tau \), denote \( \partial_{\tau}u_{\epsilon} = W \), then
\[
\partial_{\tau}W - \frac{\sigma^2}{2}\partial_{xx}W - (r - q - \frac{\sigma^2}{2})\partial_{x}W + rW + \beta'_r(\cdot)W = 0
\]
since \( u_{\epsilon}(x, 0) = \pi_{\epsilon}(k - e^{x}) \), so
\[
W(x, 0) = \partial_{\tau}u_{\epsilon}(x, 0)
\]
\[
= \frac{\sigma^2}{2}\partial_{xx}\pi_{\epsilon} + (r - q - \frac{\sigma^2}{2})\partial_{x}\pi_{\epsilon} - r\pi_{\epsilon} - \beta_{\epsilon}(0)
\]
\[
\geq r[\pi'_{\epsilon}(k - e^{x})(-e^{x}) - \pi_{\epsilon}(k - e^{x})] - \beta_{\epsilon}(0)
\]
\[
\geq -2r(k + 1) - \beta_{\epsilon}(0) \geq 0,
\]
if we take \( \beta_{\epsilon}(0) \leq -2r(k + 1) \), from minimum principle we have \( \partial_{\tau}u_{\epsilon} \geq 0 \).

**Remark:** (2.4) can be proved as well by following method:

For any \( \delta > 0 \), denote \( \overline{u}_{\epsilon}(x, \tau) = u_{\epsilon}(x, \tau + \delta) \), then \( \overline{u}_{\epsilon}(x, \tau) \) satisfies, by (2.1),
\[
\begin{cases}
\partial_{\tau}\overline{u}_{\epsilon} - \frac{\sigma^2}{2}\partial_{xx}\overline{u}_{\epsilon} - (r - q - \frac{\sigma^2}{2})\partial_{x}\overline{u}_{\epsilon} + r\overline{u}_{\epsilon} + \beta_{\epsilon}(\overline{u}_{\epsilon} - \pi_{\epsilon}(k - e^{x})) = 0, \\
\overline{u}_{\epsilon}(x, 0) = u_{\epsilon}(x, \delta) \geq \pi_{\epsilon}(k - e^{x}) = u_{\epsilon}(x, 0).
\end{cases}
\]

Applying comparison principle we know that \( \overline{u}_{\epsilon} \geq u_{\epsilon} \). It means that \( \partial_{\tau}u_{\epsilon} \geq 0 \). Thus (2.4) follows by Letting \( \epsilon \to 0 \).
Lemma 2.3.
\[ \partial_x u - \partial_x(k - e^x) \geq 0. \]  
(2.5)
i.e., \( u - (k - e^x) \) is monotonic increasing with respect to \( x \).

**Proof:** We only need to prove \( \partial_x u + e^x \geq 0 \). Denote \( v = \partial_x u \), from (2.3) we have
\[
\begin{align*}
\partial_x(v + e^x) - \frac{\sigma^2}{2} \partial_{xx}(v + e^x) - (r - q - \frac{\sigma^2}{2})\partial_x(v + e^x) + \beta_x'(\cdot)(v + e^x) \\
= \beta_x'(\cdot)\pi_x^e(k - e^x)(-e^x) + qe^x + \beta_x'(\cdot)e^x \geq qe^x \geq 0
\end{align*}
\]
and when \( \tau = 0, v + e^x = \pi_x^e(k - e^x)(-e^x) + e^x \geq 0 \).

Lemma 2.4. When \( \tau > 0 \)
\[ u(x, \tau) > 0. \]  
(2.6)

**Proof:** From
\[
\begin{align*}
\partial_x u - \frac{\sigma^2}{2} \partial_{xx}u - (r - q - \frac{\sigma^2}{2})\partial_x u + ru \geq 0, \quad (x, \tau) \in \Omega_T \\
u(x, 0) = (k - e^x)^+ \geq 0, \quad x \in \mathbb{R}
\end{align*}
\]
and strong minimum principle we know that \( u(x, \tau) > 0 (\tau > 0) \).

In the following we study the properties of free boundary. Define coincidence set
\[
\mathcal{C} = \{(x, \tau) | u(x, \tau) = (k - e^x)^+ \},
\]
noncoincidence set
\[
\mathcal{N} = \{(x, \tau) | u(x, \tau) > (k - e^x)^+ \}.
\]
Since \((k - e^x)^+ = 0\) if \( x \geq \ln k \), applying (2.6) we know that
\[
\begin{align*}
\{x \geq \ln k\} &\subset \mathcal{N} \\
\{x < \ln k\} &\supset \mathcal{C}.
\end{align*}
\]  
(2.7)
From (2.5) we can define free boundary
\[ h(\tau) = \max\{x | u(x, \tau) = (k - e^x)^+\}, \quad 0 < \tau \leq T. \]
By the definition of \( \mathcal{C} \) we see that
\[
\{x = h(\tau)\} \subset \partial \mathcal{C},
\]
and
\[ h(\tau) = \max\{x | u(x, \tau) = k - e^x\}, \quad 0 < \tau \leq T. \]

Lemma 2.5.
\[ \mathcal{C} \subset \{x \leq \ln \frac{rk}{q}\}. \]
Proof: Recalling \( x < \ln k \) on \( C \), so
\[ u = (k - e^x)^+ = k - e^x, \quad \text{on } C \]
Applying \( L u \geq 0 \) on \( C \), i.e.,
\[
-\frac{\sigma^2}{2} (-e^x) - (r - q - \frac{\sigma^2}{2})(-e^x) + r(k - e^x)
= -qe^x + rk \geq 0.
\]
thus we have
\[ x \leq \ln \frac{rk}{q} \quad \text{on } C. \]

Theorem 2.6.
\[ x_\infty \leq h(\tau) \leq \min \{ \ln k, \ln \frac{rk}{q} \}, \]
where \( x_\infty \) is the free boundary point of ODE
\[
\begin{cases}
-\frac{\sigma^2}{2} u''_\infty - (r - q - \frac{\sigma^2}{2}) u'_\infty + ru_\infty \geq 0, & u_\infty - (k - e^x)^+ \geq 0, & x \in \mathbb{R}^1, \\
-\frac{\sigma^2}{2} u''_\infty - (r - q - \frac{\sigma^2}{2}) u'_\infty + ru_\infty [u_\infty - (k - e^x)^+] = 0
\end{cases}
\quad (2.8)
\]
Proof: \( h(\tau) \leq \min \{ \ln k, \ln \frac{rk}{q} \} \) is a consequence of lemma 2.5 and (2.7).
The problem (2.8) can be solved as the following: To find \( u_\infty(x), x_\infty \) satisfying
\[
\begin{cases}
-\frac{\sigma^2}{2} u''_\infty - (r - q - \frac{\sigma^2}{2}) u'_\infty + ru_\infty = 0, & x > x_\infty, \\
u_\infty(x_\infty) = k - e^{x_\infty}, & u_\infty(+\infty) = 0 \\
u'_\infty(x_\infty) = -e^{x_\infty}
\end{cases}
\quad (2.9)
\]
this problem has an explicit formulas for \( u_\infty(x) \) and \( x_\infty \) (see the book of Jiang)
\[
u_\infty(x) = -\frac{1}{\alpha_-} \left( \frac{1}{1 - 1/\alpha_-} \right)^{1-\alpha_-} (e^x)^{\alpha_-}
\]
\[ x_\infty = \ln \frac{k}{1 - 1/\alpha_-} \]
where
\[
\alpha_- = w - \sqrt{w^2 + \frac{2r}{\sigma^2}}
\]
\[ w = \frac{-r + q + \sigma^2/2}{\sigma^2} \]
Since \( \partial_\tau u_\infty(x) = 0 \), so
\[
\begin{cases}
L u_\infty \geq 0, & u_\infty - (k - e^x)^+ \geq 0 \\
(L u_\infty)(u_\infty - (k - e^x)^+) = 0 \\
\left. u_\infty(x) \right|_{\tau=0} = u_\infty(x)
\end{cases}
\]
where
\[ \mathcal{L} = \partial_\tau - \frac{\sigma^2}{2} \partial_{xx} - (r - q - \frac{\sigma^2}{2}) \partial_x + r \]
and \( u_\infty(x) \) as a solution of parabolic obstacle problem (0.2) which initial value is itself. Since
\[ u_\infty(x)|_{\tau=0} \geq (k - e^x)^+ = u(x, 0) \]
applying the monotonicity of solution of variational inequality with respect to initial value (it is easy to prove), we have
\[ u_\infty(x) \geq u(x, \tau), \ \forall \tau \geq 0 \]
so
\[ u_\infty - (k - e^x)^+ \geq u - (k - e^x)^+ \geq 0 \]
therefore
\[ \mathcal{C}(u_\infty) \subset \mathcal{C}(u) \]
in this way we proved
\[ x_\infty \leq h(\tau). \]

**Theorem 2.7.** \( h(\tau) \) is a monotonic decreasing curve with respect to \( \tau \).

**Proof:** From lemma 2.3 we know that
\[ \partial_\tau[u - (k - e^x)^+] \geq 0, \quad (2.10) \]
Suppose \( x_0 = h(\tau_0) \), then from the definition of \( h(\tau) \),
\[ u(x, \tau_0) \quad \begin{cases} = (k - e^x)^+ = 0, & x \leq x_0 \\ > (k - e^x)^+, & x > x_0. \end{cases} \]
Applying (2.10) we know that
\[ u(x, \tau) > (k - e^x)^+, \quad x > x_0, \ \tau > \tau_0, \]
it means that \( h(\tau) \leq x_0 \) for \( \tau > \tau_0 \), hence \( h(\tau) \) is monotonic decreasing.

Since \( h(\tau) \) is monotonic on \( (0, T] \), so we can define
\[ h(0) = \lim_{\tau \to 0^+} h(\tau) \]

**Theorem 2.7.** \( h(\tau) \) is continuous on \([0, T]\) with
\[ h(0) = \min \left\{ \ln k, \ln \frac{rk}{q} \right\}. \quad (2.11) \]
If but it is not possible.

Proof: First we prove (2.11). We denote min \(\{\ln k, \ln \frac{\nu_1}{q}\} = x_0\). We know that \(h(0) \leq x_0\).

If \(h(0) < x_0\), then there is a \(\delta > 0\), such that

\[
\mathcal{L} u = 0, \quad x_0 - \delta < x < x_0, \quad 0 < \tau \leq T
\]

\[
u(x, 0) = k - e^x, \quad x_0 - \delta < x < x_0.
\]

especially

\[
u_x(x, 0) = -e^x, \quad \nu_{xx}(x, 0) = -e^x,
\]

Applying the equation at \(\tau = 0\), we have

\[
\partial_\tau \nu(x, 0) = \frac{1}{2} \sigma^2 (-e^x) + (r - q - \frac{\sigma^2}{2}) (-e^x) - r (k - e^x)
\]

\[
\sigma^2 e^x - rk < 0, \quad x_0 - \delta < x < x_0
\]

this is a contradiction with \(\partial_\tau \nu \geq 0\).

In a same way we can prove that \(h(\tau)\) is continuous on \([0, T]\).

**Theorem 2.8.** \(x = h(\tau)\) is strictly decreasing.

Proof: If \(x = h(\tau)\) has a vertical part, then on this part \(u = k - e^x, \nu_x = -e^x\), it follows that \(u_\tau = u_{\tau x} = 0\), from strong minimum principle for \(u_\tau\) we have \(u_{\tau x} > 0\), otherwise \(u_{\tau} \equiv 0\), but it is not possible.

**3. Free boundary** \(x = h(\tau) \in C^\infty (0, T)\)

**Lemma 3.1.** \(-R < a < b < R, \quad 0 < \tau_0 \leq \tau_1 < T, \quad Q = (-R, R) \times (\tau_0, T)\),

\[
\int_a^b \int_{\tau_0}^{\tau_1} \beta'_e (\nu e - \pi e) (\partial_\tau \nu e)^2 d\tau d\tau \leq C
\]

where \(\nu e\) is the solution of approximation problem (2.1), \(C\) is independent of \(e\).

Proof: Differentiating (2.1) with respect to \(\tau\), we obtain

\[
\partial_{\tau \tau} \nu e = \frac{\sigma^2}{2} \partial_{\tau x x} \nu e - (r - q - \frac{\sigma^2}{2}) \partial_{\tau x} \nu e + r \partial_{\tau} \nu e + \beta'_e (\cdot) \partial_{\tau} \nu e = 0
\] (3.1)

Suppose \(\phi(x, \tau)\) is a cut-off function in \(Q\), such that

\[
\phi \equiv 1, \quad a \leq x \leq b, \quad \tau_1 \leq \tau \leq T
\]

Multiplying (3.1) by \(\phi^2 \partial_{\tau} \nu e\), then integrating on \(Q\) reduces that

\[
\int_0^1 \frac{1}{2} \partial_{\tau} \nu e |^2 \phi^2 - \frac{\sigma^2}{2} \int_0^1 \partial_{\tau x x} \nu e \phi^2 \partial_{\tau} \nu e - (r - q - \frac{\sigma^2}{2}) \int_0^1 \partial_{\tau x} \nu e \phi^2 \partial_{\tau} \nu e
\]

\[
+ r \int_0^b |\partial_{\tau} \nu e |^2 \phi^2 + \int_0^b |\beta'_e (\cdot) |^2 (\partial_{\tau} \nu e)^2 \phi^2 = 0
\]

its first term

\[
I_1 = \frac{1}{2} \int_a^b |\partial_{\tau} \nu e |^2 \phi^2 (x, T) dx - \int_0^b |\partial_{\tau} \nu e |^2 \phi \phi_r \geq \frac{1}{2} \int_a^b |\partial_{\tau} \nu e |^2 \phi^2 (x, T) dx - C
\]
the second term

\[
I_2 = \frac{\sigma^2}{2} \int_{Q} (\partial_{x^T} u_\varepsilon)^2 \phi^2 + \frac{\sigma^2}{Q} \int_{Q} \partial_{x^T} u_\varepsilon \partial_{r^T} u_\varepsilon \phi_x
\]

\[
\geq \frac{\sigma^2}{4} \int_{Q} (\partial_{x^T} u_\varepsilon)^2 \phi^2 - C
\]

and

\[
|I_3| \leq \frac{\sigma^2}{8} \int_{Q} |\partial_{x^T} u_\varepsilon|^2 \phi^2 + C
\]

We completed the proof of lemma 3.1.

This figure is for lemma 3.1 and 3.2.

Lemma 3.2.

\[
\int_{A}^{B} \int_{\tau_1}^{T} |\partial_{r^T} u_\varepsilon|^2 dx d\tau + \int_{A}^{B} |\partial_{x^T} u_\varepsilon|^2 (x, \tau) dx \leq C
\]

where \( C \) is independent of \( \varepsilon, a < A < B < b, \tau_1 < \tau_1' < \tau < T \)

**Proof:** Multiplying (3.1) by \( \phi^2 \partial_{r^T} u_\varepsilon \), then integrating on \( Q_\tau = (a, b) \times (\tau_1, \tau) \), then

\[
\int_{Q_\tau} \phi^2 |\partial_{r^T} u_\varepsilon|^2 - \frac{\sigma^2}{2} \int_{Q_\tau} \partial_{x^T} u_\varepsilon \phi^2 \partial_{r^T} u_\varepsilon - (r - q - \frac{\sigma^2}{2}) \int_{Q_\tau} \partial_{x^T} u_\varepsilon \phi^2 \partial_{r^T} u_\varepsilon
\]

\[
+ r \int_{Q_\tau} \partial_{r} u_\varepsilon \phi^2 \partial_{r^T} u_\varepsilon + \int_{Q_\tau} \beta'_\varepsilon(\cdot) \partial_{r} u_\varepsilon \phi^2 \partial_{r^T} u_\varepsilon = 0
\]

(3.2)

where \( \phi \) is a cut-off function in \( Q_T = (a, b) \times (\tau_1, T) \) satisfying \( \phi \equiv 1 \) on \( (A, B) \times (\tau_1', T) \).

The second term in (3.2)

\[
I_2 = \frac{\sigma^2}{4} \int_{Q_\tau} \partial_{r} (\partial_{x^T} u_\varepsilon)^2 \phi^2 + \frac{\sigma^2}{2} \int_{Q_\tau} \partial_{x^T} u_\varepsilon \partial_{r^T} u_\varepsilon \phi_x
\]

\[
\geq \frac{\sigma^2}{4} \int_{a}^{b} (\partial_{x^T} u_\varepsilon)^2 \phi^2 (x, \tau) dx - \frac{1}{4} \int_{Q_\tau} \phi^2 |\partial_{r^T} u_\varepsilon|^2 - C \int_{Q_\tau} |\partial_{x^T} u_\varepsilon|^2
\]

(3.3)

the last term was estimated in lemma 3.1.

\[
|I_3|, |I_4| \leq C + \frac{1}{4} \int_{Q_\tau} \phi^2 |\partial_{r^T} u_\varepsilon|^2
\]

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\[ I_5 = \int_{Q_r} \beta'_r(\cdot) \frac{1}{2} \partial_r (\partial_r u_x)^2 \phi^2 \]
\[ = \frac{1}{2} \int_a^b \beta'_{\xi}(\cdot)(\partial_{\xi} u_x)^2 \phi^2(x, \tau) \, dx - \frac{1}{2} \int_{Q_r} \beta''_{\xi}(\cdot)(\partial_{\xi} u_x)^3 \phi^2 \, dx - \int_{Q_r} \beta'_{\xi}(\cdot)(\partial_{\xi} u_x)^2 \phi \tau \]
\[ = I_{51} + I_{52} + I_{53} \]

\[ I_{51}, I_{52} \geq 0, \]

\[ I_{53} \] was estimated in lemma 3.1 as well.

**Theorem 3.3.** \( \partial_r u \in C(\Omega_T) \), especially \( \partial_r u \) is continuous across the free boundary.

**Proof:** From lemma 3.2 we know that \( \partial_r u_x \in L^\infty(\tau'_1, T; H^1(A, B)) \)
\[ \partial_r(\partial_r u_x) \in L^2(\tau'_1, T; L^2(A, B)) \]

Applying Simon’s compact theorem, we know that \( \partial_r u \in C(\tau'_1, T; C[A, B]) \).

**Remark:** Compact sets in the space \( L^p(0, T; B) \) Ann. Mat. Appl. 146(1987), 65-96, Corollary 4:

Let \( X, Z \) and \( Y \) be Banach spaces with \( X \subset Z \subset Y \). Moreover the embedding operator \( X \to Z \) is compact. Let \( F \) be a set of functions \( [0, T] \to X \).

1. If \( F \) is bounded in \( L^p(0, T; X) \), where \( 1 \leq p < \infty \), and \( \frac{\partial F}{\partial t} = \{ \frac{\partial f}{\partial t} : f \in F \} \) is bounded in \( L^1(0, T; Y) \), then \( F \) is relatively compact in \( L^p(0, T; Z) \).
2. If \( F \) is bounded in \( L^\infty(0, T; X) \), and \( \frac{\partial F}{\partial t} \) is bounded in \( L^r(0, T; Y) \), where \( r > 1 \), then \( F \) is relatively compact in \( C([0, T]; Z) \).

**Corollary 3.4.** \( \partial_{xx} u \) is continuous up to \( x = h(\tau) \) from \( x > h(\tau) \).

**Theorem 3.5.** \( h(\tau) \in C^{3/4}(0, T] \)

**Proof:** Denote \( w(x, \tau) = u_x(x, \tau) + e^x \) then \( w(h(\tau), \tau) \equiv 0, \ \tau > 0 \), so
\[ 0 = w(h(\tau + \lambda), \tau + \lambda) - w(h(\tau), \tau) \]
\[ = w(h(\tau + \lambda), \tau + \lambda) - w(h(\tau), \tau + \lambda) + w(h(\tau), \tau + \lambda) - w(h(\tau), \tau) \]

Rewrite it as integration form
\[ \int_{h(\tau)}^{h(\tau + \lambda)} w_x(x, \tau + \lambda) \, dx + \int_{\tau}^{\tau + \lambda} w_\tau(h(\tau), \sigma) \, d\sigma = 0 \quad (3.4) \]

Let \( x = h(\tau) \) in the equation \( \partial_r u - \frac{1}{2} \sigma^2 u_{xx} - (r - q - \frac{\sigma^2}{2}) u_x + ru = 0 \), then on \( x = h(\tau) \)
\[ \frac{1}{2} \sigma^2 u_{xx} = -(r - q - \frac{\sigma^2}{2})(-e^x) + r(k - e^x) \]
\[ = \frac{\sigma^2}{2} e^x - qe^x + rk, \quad x = h(\tau), \]

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\[ w_x(h(\tau), \tau) = (u_{xx} + e^x) \bigg|_{x=h(\tau)} = \frac{2}{\sigma^2} (rk - qe^x) \bigg|_{x=h(\tau)} > 0. \]

If \( h(\tau + \lambda) < x < h(\tau) \),
\[ w_x(x, \tau + \lambda) = w_x(h(\tau), \tau) + w_x(x, \tau + \lambda) - w_x(h(\tau), \tau) = w_x(h(\tau), \tau) + o(1) \]
where \( o(1) \to 0 \) if \( \lambda \to 0 \), hence
\[
\int_{h(\tau)}^{h(\tau+\lambda)} w_x(x, \tau + \lambda) dx = \frac{2}{\sigma^2} (rk - qe^{h(\tau)})(h(\tau + \lambda) - h(\tau))(1 + o(1))
\]
\[
\geq \frac{1}{\sigma^2} (rk - qe^{h(\tau)})(h(\tau + \lambda) - h(\tau)) \quad (3.5)
\]
by \( 1 + o(1) \geq \frac{1}{2} \) if \( \lambda \) is small enough.

On the other hand
\[ w_x(h(\tau), t) = w_x(h(\tau) + \eta, t) - \int_{h(\tau)}^{h(\tau)+\eta} w_{x\tau}(x, t) dx, \quad \tau < t < \tau + \lambda \quad (3.6) \]
since
\[ u_{\tau\tau} - \frac{1}{2} \sigma^2 u_{x\tau\tau} - (r - q - \frac{\sigma^2}{2}) u_{x\tau} + ru_{\tau} = 0 \]
hence
\[ w_{x\tau} = u_{x\tau\tau} = \frac{2}{\sigma^2} [u_{\tau\tau} - (r - q - \frac{\sigma^2}{2}) u_{x\tau} + ru_{\tau}] \]
substituting it into (3.6), we obtain
\[ w_x(h(\tau), t) = w_x(h(\tau) + \eta, t) - \frac{2}{\sigma^2} \int_{h(\tau)}^{h(\tau)+\eta} [u_{\tau\tau} - (r - q - \frac{\sigma^2}{2}) u_{x\tau} + ru_{\tau}](x, t) dx \]
therefore
\[
\int_{\tau}^{\tau+\lambda} w_x(h(\tau), t) dt = \int_{\tau}^{\tau+\lambda} w_x(h(\tau) + \eta, t) dt - \frac{2}{\sigma^2} \int_{\tau}^{\tau+\lambda} \int_{h(\tau)}^{h(\tau)+\eta} [u_{\tau\tau} - (r - q - \frac{\sigma^2}{2}) u_{x\tau} + ru_{\tau}](x, t) dx dt \quad (3.7)
\]
substituting (3.5), (3.7) into (3.4), and integrating with respect to \( \eta \) on \( (0, m) \), we obtain
\[
\frac{1}{\sigma^2} (rk - e^{h(\tau)}) m |h(\tau + \lambda) - h(\tau)|
\]
\[
\leq \int_{\tau}^{\tau+\lambda} \int_{0}^{m} |w_x(h(\tau) + \eta, t)| d\eta dt
\]
\[
+ \frac{2m}{\sigma^2} \int_{\tau}^{\tau+\lambda} \int_{h(\tau)}^{h(\tau)+m} |u_{\tau\tau} - (r - q - \frac{\sigma^2}{2}) u_{x\tau} + ru_{\tau}|(x, t) dx dt
\]
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i.e.,
\[ m|h(\tau + \lambda) - h(\tau)| \leq C\lambda m^{1/2} + Cm(\lambda m)^{1/2} \]

Taking \( m = \lambda^{1/2} \), then
\[ |h(\tau + \lambda) - h(\tau)| \leq C\lambda^{3/4} \]

**Lemma 3.6.** If
\[
\partial_\tau w - \partial_{xx} w = 0, \quad \sigma(\tau) < x < S(\tau), \quad 0 < \tau < T
\]

Suppose (1). \( S(\tau) \in C^\alpha, \alpha > \frac{1}{2} \).
(2). \( w \) is continuous up to \( x = S(\tau) \), and \( w(S(\tau), \tau) = 0 \),
then \( w_x \) is continuous up to \( x = S(\tau) \).


**Theorem 3.7.** \( h(t) \in C^1(0, T] \).
**Proof:** It follows, from lemma 3.6 and theorem 3.5, that \( \partial_{xx} u \) is continuous up to \( x = h(t) \).
Considering \( u_x(h(\tau), \tau) = -e^{h(\tau)} \) we have that \( u_{xx} h' + u_{x\tau} = -e^h h' \), hence
\[
h'(\tau) = -\frac{u_{xx}}{u_{xx} + e^{h(\tau)}} = -\frac{\sigma^2}{2} \frac{u_{xx}(h(\tau), \tau)}{r k - q e^{h(\tau)}}
\]

therefore \( h \in C^1(0, T] \).

**Theorem 3.8.** \( h(\tau) \in C^{\infty}(0, T] \).
**Proof:** Applying a transformation of coordinates
\[
y = x - h(\tau), \quad \tau = \tau,
\]
and denote \( v(y, \tau) = u_\tau(x, \tau) \), then
\[
u_{\tau\tau} = v_\tau - h'(\tau)v_y
\]
\[
u_{x\tau} = v_y
\]
\[
u_{xx\tau} = v_{yy}
\]
The domain \( \{ x > h(\tau), \ 0 < \tau \leq T \} \) is changed to \( \{ (y, \tau)|y > 0, 0 < \tau \leq T \} \), \( x = h(\tau) \) becomes \( y = 0 \), \( u_\tau(x, \tau) \) satisfying
\[
\partial_\tau(u_\tau) - \frac{\sigma^2}{2} \partial_{xx}(u_\tau) - (r - q - \frac{\sigma^2}{2}) \partial_x(u_\tau) + ru_\tau = 0
\]

therefore \( v(y, \tau) \) satisfies
\[
\begin{cases}
\partial_\tau v - \frac{\sigma^2}{2} v_{yy} - (r - q - \frac{\sigma^2}{2} + h'(\tau))v_y + rv = 0, \quad y > 0, \ 0 < \tau \leq T \\
v = 0 \text{ on } y = 0 \\
v(y, 0) = u_\tau(y, 0)
\end{cases}
\]

(3.8)

\[
h'(\tau) = -\frac{\sigma^2}{2} \frac{\partial_y v(0, \tau)}{r k - q e^{h(\tau)}}
\]

(3.9)
Applying $W^{2,1}_p$ estimate to the system (3.8) we know that $|v|_{W^{2,1}_p(Q)} \leq C$, where

$$Q = (0, 1) \times (\delta, T)$$

It is reduced, by embedding theorem, that $v \in C^{1+\alpha,(1+\alpha)/2}(\overline{Q})$, $\forall \alpha > 0$. So $\partial_y v \in C^{\alpha,\alpha/2}(\overline{Q})$. Applying

$$\begin{cases}
(3.9) \\
h \in C^1 \\
\partial_y v \in C^{\alpha,\alpha/2}
\end{cases}$$

we know that $h'(\tau) \in C^{\alpha/2}(0, T]$.

Backing to the equation (3.8), now we have $v \in C^{1+\alpha,(1+\alpha)/2}(\overline{Q})$, so $\partial_y v \in C^{1+\alpha,(1+\alpha)/2}(\overline{Q})$. Again applying

$$\begin{cases}
(3.9) \\
h \in C^{1+\alpha/2} \\
\partial_y v \in C^{1+\alpha,(1+\alpha)/2}
\end{cases}$$

we obtain that $h' \in C^{(1+\alpha)/2}(0, T]$. At last by a bootstrap argument, $h(\tau) \in C^\infty(0, T]$. 

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