GRÖBNER–SHIRSHOV BASES AND EMBEDDINGS OF ALGEBRAS

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In this paper, by using Gröbner–Shirshov bases, we show that in the following classes, each (respectively, countably generated) algebra can be embedded into a simple (respectively, two-generated) algebra: associative differential algebras, associative \( \Omega \)-algebras, associative \( \lambda \)-differential algebras. We show that in the following classes, each countably generated algebra over a countable field \( k \) can be embedded into a simple two-generated algebra: associative algebras, semigroups, Lie algebras, associative differential algebras, associative \( \Omega \)-algebras, associative \( \lambda \)-differential algebras. We give another proofs of the well known theorems: each countably generated group (respectively, associative algebra, semigroup, Lie algebra) can be embedded into a two-generated group (respectively, associative algebra, semigroup, Lie algebra).

Keywords: Gröbner–Shirshov basis; group; associative algebra; Lie algebra; associative differential algebra; associative \( \Omega \)-algebra.

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1. Introduction

G. Higman, B. H. Neumann and H. Neumann [26] proved that any countable group is embeddable into a 2-generated group. It means that the basic rank of variety of groups is equal to two. In contrast, for example, the basic ranks of varieties of alternative and Malcev algebras are equal to infinity (I. P. Shestakov [37]): there is no such $n$, that a countably generated alternative (Malcev) algebra can be embeddable into $n$-generated alternative (Malcev) algebra. Even more, for any $n \geq 1$, there exists an alternative (Malcev) algebra generated by $n + 1$ elements which cannot be embedded into an $n$-generated alternative (Malcev) algebra (V. T. Filippov [20, 21]). For Jordan algebras, it is known that the basic rank is bigger than 2, since any 2-generated Jordan algebra is special (A. I. Shirshov [39, 41]), but there exist (even finitely dimensional) non-special Jordan algebras (A. A. Albert [1]). A. I. Malcev [34] proved that any countably generated associative algebra is embeddable into a 2-generated associative algebra. A. I. Shirshov [38, 41] proved the same result for Lie algebras and T. Evans [19] proved the same result for semigroup.

The first example of finitely generated infinite simple group was constructed by G. Higman [25]. Later P. Hall [24] proved that any group is embeddable into a simple group which is generated by three prescribed subgroups with some cardinality conditions. In particular, any countably generated group is embeddable into a simple 3-generated group.

B. Neumann [35] proved that any non-associative algebra is embeddable into a non-associative division algebra such that any equation $ax = b$, $xa = b$, $a \neq 0$ has a solution in the latter. Any division algebra is simple. P. M. Cohn [18] proved that any associative ring without zero divisors is embeddable into a simple associative ring without zero divisors such that any equation $ax - xa = b$, $a \neq 0$, has a solution in the latter. L. A. Skornyakov [42] proved that any non-associative algebra without zero divisors is embeddable into a non-associative division algebra without zero divisors. I. S. Ivanov [28, 29] prove the same result for $\Omega$-algebras (see also A. G. Kurosh [31]). P. M. Cohn [18] proved that any Lie algebra is embeddable into a division Lie algebra. E. G. Shutov [43] and L. A. Bokut [7] proved that any semigroup is embeddable into a simple semigroup, and L. A. Bokut [9] proved that any associative algebra is embeddable into a simple associative algebra such that any equation $xy = b$, $a \neq 0$ is solvable in the latter. L. A. Bokut [5, 6] proved that any Lie (respectively, non-associative, commutative, anti-commutative) algebra $A$ is embeddable into an algebraically closed (in particular simple) Lie (respectively, non-associative, commutative, anti-commutative) algebra $B$ such that any equation $f(x_1, \ldots, x_n) = 0$ with coefficient in $B$ has a solution in $B$ (an equation over $B$ is an element of a free product of $B$ with a corresponding free algebra $k(X)$). L. A. Bokut [9, 10, 13] proved that any associative (Lie) algebra is embeddable into a simple associative (algebraically closed Lie) algebra which is a sum of 4 prescribed (Lie) subalgebras with some cardinality conditions. In particular any countable associative (Lie) algebra is embeddable into a simple finitely generated associative
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Yu. Bahturin and A. Olshanskii [2] prove that each finitely generated associative (Lie) algebra can be embedded into a simple 2-generated algebra without distortion.

In this paper, by using Gröbner–Shirshov bases and some ideas from [9, 10], we prove that in the following classes, each (respectively, countably generated) algebra can be embedded into a simple (respectively, two-generated) algebra: associative differential algebras, associative Ω-algebras, associative \(\lambda\)-differential algebras. We show that in the following classes, each countably generated algebra over a countable field \(k\) can be embedded into a simple two-generated algebra: associative algebras, semigroups, Lie algebras, associative differential algebras, associative Ω-algebras, associative \(\lambda\)-differential algebras. We give another proofs of Higman–Neumann–Neumann’s and Shirshov’s results mentioned above.

We systematically use Gröbner–Shirshov bases theory for associative algebras, Lie algebras, associative Ω-algebras, associative differential algebras, see [40, 41, 12, 17].

2. Preliminaries

We first cite some concepts and results from the literature [8, 9, 40, 41] which are related to Gröbner–Shirshov bases for associative algebras.

Let \(X\) be a set and \(k\) a field. Throughout this paper, we denote by \(k\langle X\rangle\) the free associative algebra over \(k\) generated by \(X\), by \(X^*\) the free monoid generated by \(X\) and by \(N\) the set of natural numbers.

A well ordering \(<\) on \(X^*\) is called monomial if for \(u, v \in X^*\), we have
\[ u < v \Rightarrow w|_u < w|_v \quad \text{for all } w \in X^*, \]
where \(w|_u = w|x_i\mapsto u\), \(w|_v = w|x_i\mapsto v\), and \(x_i\)'s are the same individuality of the letter \(x_i \in X\) in \(w\).

A standard example of monomial ordering on \(X^*\) is the deg-lex ordering which first compare two words by degree and then by comparing them lexicographically, where \(X\) is a well-ordered set.

Let \(X^*\) be a set with a monomial ordering \(<\). Then, for any polynomial \(f \in k\langle X\rangle\), \(f\) has the leading word \(\overrightarrow{f}\). We call \(f\) monic if the coefficient of \(\overrightarrow{f}\) is 1. By \(\deg(f)\) we denote the degree of \(\overrightarrow{f}\).

Let \(f, g \in k\langle X\rangle\) be two monic polynomials and \(w \in X^*\). If \(w = \overrightarrow{f}b = a\overrightarrow{g}\) for some \(a, b \in X^*\) such that \(\deg(\overrightarrow{f}) + \deg(\overrightarrow{g}) > \deg(w)\), then \((f, g)_w = fb - ag\) is called the intersection composition of \(f, g\) relative to \(w\). If \(w = \overrightarrow{f} = a\overrightarrow{g}\) for some \(a, b \in X^*\), then \((f, g)_w = f - agb\) is called the inclusion composition of \(f, g\) relative to \(w\). In \((f, g)_w\), \(w\) is called the ambiguity of the composition.

Let \(S \subset k\langle X\rangle\) be a monic set. A composition \((f, g)_w\) is called trivial modulo \((S, w)\), denoted by
\[ (f, g)_w \equiv 0 \mod(S, w) \]
if \((f, g)_w = \sum \alpha_i a_i s_i b_i\), where every \(\alpha_i \in k, s_i \in S, a_i, b_i \in X^*\), and \(a_i b_i < w\).
Recall that $S$ is a Gröbner–Shirshov basis in $k\langle X \rangle$ if any composition of polynomials from $S$ is trivial modulo $S$.

The following lemma was first proved by Shirshov [40, 41] for free Lie algebras (with deg-lex ordering) (see also Bokut [8]). Bokut [9] specialized the approach of Shirshov to associative algebras (see also Bergman [4]). For commutative polynomials, this lemma is known as Buchberger’s Theorem (see [14, 15]).

**Lemma 2.1 (Composition-Diamond lemma for associative algebras).** Let $k$ be a field, $A = k\langle X \mid S \rangle = k\langle X \rangle/\text{Id}(S)$ and $< \text{a monomial ordering on } X^*$, where $\text{Id}(S)$ is the ideal of $k\langle X \rangle$ generated by $S$. Then the following statements are equivalent:

1. $S$ is a Gröbner–Shirshov basis in $k\langle X \rangle$.
2. $f \in \text{Id}(S) \Rightarrow f = \alpha sb$ for some $s \in S$ and $a, b \in X^*$.
3. $\text{Irr}(S) = \{u \in X^* \mid u \neq \alpha sb, s \in S, a, b \in X^*\}$ is a $k$-basis of the algebra $A = k\langle X \mid S \rangle = k\langle X \rangle/\text{Id}(S)$.

If a subset $S$ of $k\langle X \rangle$ is not a Gröbner–Shirshov basis then one can add all nontrivial compositions of polynomials of $S$ to $S$. Continue this process repeatedly, we finally obtain a Gröbner–Shirshov basis $S^{\text{comp}}$ that contains $S$. Such a process is called Shirshov algorithm.

Let $A = \text{sgp}(X \mid S)$ be a semigroup presentation. Then $S$ is also a subset of $k\langle X \rangle$ and we can find Gröbner–Shirshov basis $S^{\text{comp}}$. We also call $S^{\text{comp}}$ a Gröbner–Shirshov basis of $A$. $\text{Irr}(S^{\text{comp}}) = \{u \in X^* \mid u \neq \alpha \text{fb}, a, b \in X^*, f \in S^{\text{comp}}\}$ is a $k$-basis of $k\langle X \mid S \rangle$ which is also the set of all normal words of $A$.

The following lemma is well known and can be easily proved.

**Lemma 2.2.** Let $k$ be a field, $S \subset k\langle X \rangle$. Then for any $f \in k\langle X \rangle$, $f$ can be expressed as $f = \sum_{u_i \in \text{Irr}(S), u_i \leq f} \alpha_i u_i + \sum \beta_j a_j b_j, s_j \in S$, where $\alpha_i, \beta_j \in k, a_j, b_j \in X^*$.

The analogous lemma is valid for the free Lie algebra Lie($X$) (see, for example, [11]).

**Lemma 2.3.** Let $k$ be a field, $S \subset \text{Lie}(X)$. Then for any $f \in \text{Lie}(X)$, $f$ can be expressed as $f = \sum_{[u] \in \text{Irr}(S), [u] \leq f} \alpha_i [u_i] + \sum \beta_j [a_j b_j], s_j \in S$, and $\text{Irr}(S) = \{[u] \mid [u] \text{ is a non-associative Lyndon–Shirshov word on } X, u \neq \alpha sb, s \in S, a, b \in X^*\}$. 

### 3. Associative Algebras, Groups and Semigroups

In this section we give another proofs for the following theorems mentioned in the introduction: every countably generated group (respectively, associative algebra, semigroup) can be embedded into a two-generated group (respectively, associative algebra, semigroup). Even more, we prove the following theorems. (i) Every countably generated associative algebra over a countable field $k$ can be embedded into a
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simple two-generated associative algebra. (ii) Every countably generated semigroup can be embedded into a (0-)simple two-generated semigroup.

In this section, all the algebras we mention contain units.

In 1949, G. Higman, B. H. Neumann and H. Neumann [26] prove that every countable group can be embedded into a two-generated group. Now we give another proof for this theorem.

**Theorem 3.1 (G. Higman, B. H. Neumann and H. Neumann).** Every countable group can be embedded into a two-generated group.

**Proof.** We may assume that the group $G = \{g_0 = 1, g_1, g_2, g_3, \ldots \}$ and for any $g_i, g_j \in G$, $[g_i, g_j]$ means the product of $g_i$ and $g_j$. Let

\[ H = gp(G \setminus \{g_0\}, a, b, t \mid g_ig_j = [g_j, g_k], at = tb, b^{-1}abt = tg_ia^{-1}ba^i, i, j, k \in N). \]

G. Higman, B. H. Neumann and H. Neumann [26] (see also [33]) prove that $G$ can be embedded into $H$. Now, we use the Composition-Diamond lemma, i.e., Lemma 2.1 to reprove this theorem.

Clearly, $H$ can also be expressed as

\[ H = gp(G \setminus \{g_0\}, a, b, t \mid S), \]

where $S$ consists of the following polynomials ($\varepsilon = \pm 1, i, j, k \in N$):

1. $g_jg_k = [g_j, g_k]$
2. $a^\varepsilon t = tb^\varepsilon$
3. $b^\varepsilon t^{-1} = t^{-1}a^\varepsilon$
4. $abt = b^\varepsilon t g_ia^{-i}ba^i$
5. $a^{-1}b^{-1}t = b^{-1}(g_i a^{-i}ba^i)^{-1}$
6. $ba^{-1}t = a^\varepsilon g_i^{-1}t^{-1}b^{-1}ab^i$
7. $b^{-1}a^{-1}g_i^{-1}t^{-1} = a^\varepsilon t^{-1}b^{-1}a^{-1}b^i$
8. $a^\varepsilon a^{-\varepsilon} = b^\varepsilon b^{-\varepsilon} = t^\varepsilon t^{-\varepsilon} = 1.$

We order $\{g_i, a^{\pm 1}, b^{\pm 1}\}^*$ by deg-lex ordering with $g_i < a < a^{-1} < b < b^{-1}$. Denote by $X = \{g_i, a^{\pm 1}, b^{\pm 1}, t^{\pm 1}\}$. For any $u \in X^*$, $u$ can be uniquely expressed without brackets as

\[ u = u_0 t^{\varepsilon_1}u_1 t^{\varepsilon_2}u_2 \cdots t^{\varepsilon_n}u_n, \]

where $u_i \in \{g_i, a^{\pm 1}, b^{\pm 1}\}^*, n \geq 0, \varepsilon_i = \pm 1$. Denote by

\[ wt(u) = (n, u_0, t^{\varepsilon_1}, u_1, t^{\varepsilon_2}, u_2, \ldots, t^{\varepsilon_n}, u_n). \]

Then, we order $X^*$ as follows: for any $u, v \in X^*$

\[ u > v \iff wt(u) > wt(v) \text{ lexicographically}, \]

where $t > t^{-1}$. With this ordering, we can check that $S$ is a Gröbner–Shirshov basis in the free associative algebra $k(X)$. By Lemma 2.1, the group $G$ can be embedded into $H$ which is generated by $\{a, t\}$.

A. I. Malcev [34] proved that any countably generated associative algebra is embeddable into a two-generated associative algebra, and T. Evans [19] proved
that every countably generated semigroup can be embedded into a two-generated semigroup. Now, by applying Lemma 2.1, we give another proofs of this two embedding theorems.

**Theorem 3.2 (A. I. Malcev).** Every countably generated associative algebra can be embedded into a two-generated associative algebra.

**Proof.** We may assume that \( A = k\langle X \mid S \rangle \) is an associative algebra generated by \( X \) with relations \( S \), where \( X = \{ x_i, i = 1, 2, \ldots \} \). By Shirshov algorithm, we can assume that \( S \) is a Gröbner–Shirshov basis in the free associative algebra \( k\langle X \rangle \) with deg-lex ordering on \( X^* \). Let \( H = k\langle X, a, b \mid S, aabab = x_i, i = 1, 2, \ldots \rangle \).

We can check that \( \{ S, aabab = x_i, i = 1, 2, \ldots \} \) is a Gröbner–Shirshov basis in \( k\langle X, a, b \rangle \) with deg-lex ordering on \( (X \cup \{ a, b \})^* \) where \( a > b > x, x \in X \) since there are no new compositions. By Lemma 2.1, \( A \) can be embedded into \( H \) which is generated by \( \{ a, b \} \).

By the proof of Theorem 3.2, we have immediately the following corollary.

**Corollary 3.3 (T. Evans).** Every countably generated semigroup can be embedded into a two-generated semigroup.

**Theorem 3.4.** Every countably generated associative algebra over a countable field \( k \) can be embedded into a simple two-generated associative algebra.

**Proof.** Let \( A \) be a countably generated associative algebra over a countable field \( k \). We may assume that \( A \) has a countable \( k \)-basis \( \{ 1 \} \cup X_0 \), where \( X_0 = \{ x_i \mid i = 1, 2, \ldots \} \) and 1 is the unit of \( A \). Then \( A \) can be expressed as \( A = k\langle X_0 \mid x_ix_j = x_jx_i, i, j \in N \rangle \), where \( \{ x_i, x_j \} \) is a linear combination of \( x_t, x_i \in X_0 \).

Let \( A_0 = k\langle X_0 \rangle, A_0^+ = A_0 \setminus \{ 0 \} \) and fix the bijection

\[
(A_0^+, A_0^+) \leftrightarrow \{ (x_m^{(1)}, y_m^{(1)}), m \in N \}.
\]

Let \( X_1 = X_0 \cup \{ x_m^{(1)}, y_m^{(1)}, a, b \mid m \in N \} \), \( A_1 = k\langle X_1 \rangle, A_1^+ = A_1 \setminus \{ 0 \} \) and fix the bijection

\[
(A_1^+, A_1^+) \leftrightarrow \{ (x_m^{(2)}, y_m^{(2)}), m \in N \}.
\]

\[
\vdots
\]

Let \( X_{n+1} = X_n \cup \{ x_m^{(n+1)}, y_m^{(n+1)} \mid m \in N \}, n \geq 1 \), \( A_{n+1} = k\langle X_{n+1} \rangle, A_{n+1}^+ = A_{n+1} \setminus \{ 0 \} \) and fix the bijection

\[
(A_{n+1}^+, A_{n+1}^+) \leftrightarrow \{ (x_m^{(n+2)}, y_m^{(n+2)}), m \in N \}.
\]

\[
\vdots
\]
Consider the chain of the free associative algebras

\[ A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots. \]

Let \( X = \bigcup_{n=0}^{\infty} X_n \). Then \( k(X) = \bigcup_{n=0}^{\infty} A_n \).

Now, define the desired algebra \( \mathcal{A} \). Take the set \( X \) as the set of the generators for this algebra and take the following relations as one part of the relations for this algebra

\[
\begin{align*}
  x_i x_j = \{x_i, x_j\}, & \quad i, j \in \mathbb{N} \quad (1) \\
  aa(ab)^n b^{2m+1} ab = x_{(n)}^m, & \quad m, n \in \mathbb{N} \quad (2) \\
  aa(ab)^n b^{2m} ab = y_n^m, & \quad m, n \in \mathbb{N} \quad (3) \\
  aabbab = x_1. & \quad (4)
\end{align*}
\]

Before we introduce the another part of the relations on \( \mathcal{A} \), let us define canonical words of the algebra \( A_n \), \( n \geq 0 \). A word in \( X_0 \) without subwords that are the leading terms of (1) is called a canonical word of \( A_0 \). A word in \( X_1 \) without subwords that are the leading terms of (1)–(4) and without subwords of the form

\[
(x_n^m)^{\deg(g(0)) + 1} f_g^{(0)} y_n^m^{(1)},
\]

where \((x_n^m, y_n^m) \mapsto (f_g^{(0)}, g^{(0)}) \in (A^+_0, A^+_0)\) such that \( f^{(0)}, g^{(0)} \) are nonzero linear combination of canonical words of \( A_0 \), is called a canonical word of \( A_1 \). Suppose that we have defined canonical word of \( A_k, k < n \). A word in \( X_n \) without subwords that are the leading terms of (1)–(4) and without subwords of the form

\[
(x_n^m)^{\deg(g^{(k)}) + 1} f_g^{(k)} y_n^m^{(k+1)},
\]

where \((x_n^m, y_n^m) \mapsto (f_g^{(k)}, g^{(k)}) \in (A^+_k, A^+_k)\) such that \( f^{(k)}, g^{(k)} \) are nonzero linear combination of canonical words of \( A_k \), is called a canonical word of \( A_n \).

Then the another part of the relations on \( \mathcal{A} \) are the following:

\[
(x_{(n)}^m)^{\deg(g^{(n-1)}) + 1} f_{g^{(n-1)}}^{(n-1)} y_{(n)}^m - g^{(n-1)} = 0, \quad m, n \in \mathbb{N} \quad (5)
\]

where \((x_{(n)}^m, y_{(n)}^m) \mapsto (f_{g^{(n-1)}}, g^{(n-1)}) \in (A^+_{n-1}, A^+_{n-1})\) such that \( f^{(n-1)}, g^{(n-1)} \) are nonzero linear combination of canonical words of \( A_{n-1} \).

By Lemma 2.2, we have that in \( \mathcal{A} \) every element can be expressed as linear combination of canonical words.

Denote by \( S \) the set constituted by the relations (1)–(5). We can see that \( S \) is a Gröbner–Shirshov basis in \( k(X) \) with deg-lex ordering on \( X^* \) since in \( S \) there are no compositions except for the ambiguity \( x_i x_j x_k \) which is a trivial case. By Lemma 2.1, \( \mathcal{A} \) can be embedded into \( \mathcal{A} \). By (5), \( \mathcal{A} \) is a simple algebra. By (2)–(5), \( \mathcal{A} \) is generated by \( \{a, b\} \). \( \blacksquare \)
A semigroup $S$ without zero is called simple if it has no proper ideals. A semigroup $S$ with zero is called 0-simple if \{0\} and $S$ are its only ideals, and $S^2 \neq \{0\}$.

**Lemma 3.5** ([27]). A semigroup $S$ with 0 is 0-simple if and only if $SaS = S$ for every $a \neq 0$ in $S$. A semigroup $S$ without 0 is simple if and only if $SaS = S$ for every $a$ in $S$.

The following theorem follows from the proof of Theorem 3.4.

**Theorem 3.6.** Every countably generated semigroup can be embedded into a simple two-generated semigroup.

**Remark.** Let $S$ be a simple semigroup. Then the semigroup $S^0$ with 0 attached is a 0-simple semigroup. Therefore, by Theorem 3.6, each countably generated semigroup can be embedded into a 0-simple two-generated semigroup.

### 4. Lie Algebras

In this section, we give another proof of the Shirshov’s theorem: every countably generated Lie algebra can be embedded into a two-generated Lie algebra. And even more, we show that every countably generated Lie algebra over a countable field $k$ can be embedded into a simple two-generated Lie algebra.

We start with the Lyndon–Shirshov associative words.

Let $X = \{x_i | i \in I\}$ be a well-ordered set with $x_i > x_p$ if $i > p$ for any $i, p \in I$. We order $X^*$ by the lexicographical ordering.

**Definition 4.1** ([32, 38, 41], see [11, 44]). Let $u \in X^*$ and $u \neq 1$. Then $u$ is called an **ALSW** (associative Lyndon–Shirshov word) if

$$(\forall v, w \in X^*, v, w \neq 1) \quad u = vw \Rightarrow vw > wv.$$ 

**Definition 4.2** ([16, 38, 41], see [11, 44]). A non-associative word $(u)$ in $X$ is called a **NLSW** (non-associative Lyndon–Shirshov word) if

(i) $u$ is an **ALSW**,  
(ii) if $(u) = ((v)(w))$, then both $(v)$ and $(w)$ are **NLSW**’s,  
(iii) in (ii) if $(v) = ((v_1)(v_2))$, then $v_2 \leq w$ in $X^*$.

**Lemma 4.3** ([16, 38, 41], see [11, 44]). Let $u$ be an **ALSW**. Then there exists a unique bracketing way such that $(u)$ is a **NLSW**.

Let $X^{**}$ be the set of all non-associative words $(u)$ in $X$. If $(u)$ is a **NLSW**, then we denote it by $[u]$.

**Lemma 4.4** ([16, 38, 41], see [11, 44]). **NLSW**’s forms a linear basis of $\text{Lie}(X)$, the free Lie algebra generated by $X$. 

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Composition-Diamond lemma for free Lie algebras (with deg-lex ordering) is given in [40, 41] (see also [11]). By applying this lemma, we give the following theorem.

**Theorem 4.5 (A. I. Shirshov).** Every countably generated Lie algebra can be embedded into a two-generated Lie algebra.

**Proof.** We may assume that \( L = \text{Lie}(X \mid S) \) is a Lie algebra generated by \( X \) with relations \( S \), where \( X = \{x_i, i = 1, 2, \ldots\} \). By Shirshov algorithm, we can assume that \( S \) is a Gröbner–Shirshov basis in the free Lie algebra \( \text{Lie}(X) \) on deg-lex ordering. Let \( H = \text{Lie}(X, a, b \mid S, \{aab\} = x_i, i = 1, 2, \ldots) \).

We can check that \( \{S, \{aab\} = x_i, i = 1, 2, \ldots\} \) is a Gröbner–Shirshov basis of \( \text{Lie}(X, a, b) \) on deg-lex ordering with \( a > b > x_i \) since there are no new compositions. By the Composition-Diamond lemma for Lie algebras, \( L \) can be embedded into \( H \) which is generated by \( \{a, b\} \).

**Theorem 4.6.** Every countably generated Lie algebra over a countable field \( k \) can be embedded into a simple two-generated Lie algebra.

**Proof.** Let \( L \) be a countably generated Lie algebra over a countable field \( k \). We may assume that \( L \) has a countable \( k \)-basis \( X_0 = \{x_i \mid i = 1, 2, \ldots\} \). Then \( L \) can be expressed as \( L = \text{Lie}(X_0 \mid x_i x_j) = \{x_i, x_j \mid i, j \in \mathbb{N}\} \).

Let \( L_0 = \text{Lie}(X_0), L_0^+ = L_0 \setminus \{0\} \) and fix the bijection \( (L_0^+, L_0^+) \leftrightarrow \{(x_m^{(1)}, y_m^{(1)}) \mid m \in \mathbb{N}\} \).

Let \( X_1 = X_0 \cup \{x_m^{(1)}, y_m^{(1)} \mid m \in \mathbb{N}\} \), \( L_1 = \text{Lie}(X_1), L_1^+ = L_1 \setminus \{0\} \) and fix the bijection \( (L_1^+, L_1^+) \leftrightarrow \{(x_m^{(2)}, y_m^{(2)}) \mid m \in \mathbb{N}\} \).

... Let \( X_{n+1} = X_n \cup \{x_m^{(n+1)}, y_m^{(n+1)} \mid m \in \mathbb{N}\} \), \( n \geq 1 \), \( L_{n+1} = \text{Lie}(X_{n+1}) \), \( L_{n+1}^+ = L_{n+1} \setminus \{0\} \) and fix the bijection \( (L_{n+1}^+, L_{n+1}^+) \leftrightarrow \{(x_m^{(n+2)}, y_m^{(n+2)}) \mid m \in \mathbb{N}\} \).
Consider the chain of the free Lie algebras

\[ L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_n \subset \cdots. \]

Let \( X = \bigcup_{n=0}^{\infty} X_n \). Then \( \text{Lie}(X) = \bigcup_{n=0}^{\infty} L_n \).

Now, define the desired Lie algebra \( \mathcal{L} \). Take the set \( X \) as the set of the generators for this algebra and take the following relations as one part of the relations for this algebra

\[ [x_i, x_j] = \{x_i, x_j\}, \quad i, j \in N \] (6)

\[ [aa(ab)^n b^{2m+1} ab] = x_m^{(n)}, \quad m, n \in N \] (7)

\[ [aa(ab)^n b^{2m} ab] = y_m^{(n)}, \quad m, n \in N \] (8)

\[ [aabbab] = x_1. \] (9)

Before we introduce the another part of the relations on \( \mathcal{L} \), let us define canonical words of the Lie algebra \( L_n \), \( n \geq 0 \). A NLSW \([u]\) in \( X_0 \) where \( u \) without subwords that are the leading terms of (6) is called a canonical word of \( L_0 \). A NLSW \([u]\) in \( X_1 \) where \( u \) without subwords that are the leading terms of (6)–(9) and without subwords of the form

\[ x_m^{(1)} f^{(0)} x_m^{(1)} (y_m^{(1)})^{\deg(g^{(0)}) + 1}, \]

where \( (x_m^{(1)}, y_m^{(1)}) \leftrightarrow (f^{(0)}, g^{(0)}) \in (L_0^+, L_0^+) \) such that \( f^{(0)}, g^{(0)} \) are nonzero linear combination of canonical words of \( L_0 \), is called a canonical word of \( L_1 \). Suppose that we have defined canonical word of \( L_k \), \( k < n \). A NLSW \([u]\) in \( X_n \) where \( u \) without subwords that are the leading terms of (6)–(9) and without subwords of the form

\[ x_m^{(k+1)} f^{(k)} x_m^{(k+1)} (y_m^{(k)})^{\deg(g^{(k)}) + 1}, \]

where \( (x_m^{(k+1)}, y_m^{(k+1)}) \leftrightarrow (f^{(k)}, g^{(k)}) \in (L_k^+, L_k^+) \) such that \( f^{(k)}, g^{(k)} \) are nonzero linear combination of canonical words of \( L_k \), is called a canonical word of \( L_n \).

Then the another part of the relations on \( \mathcal{L} \) are the following:

\[ (x_m^{(n)} f^{(n-1)}) (y_m^{(n)})^{\deg(g^{(n-1)}) + 1} - g^{(n-1)} = 0, \quad m, n \in N \] (10)

where \( (x_m^{(n)}, y_m^{(n)}) \leftrightarrow (f^{(n-1)}, g^{(n-1)}) \in (L_{n-1}^+, L_{n-1}^+) \) such that \( f^{(n-1)}, g^{(n-1)} \) are nonzero linear combination of canonical words of \( L_{n-1} \).

By Lemma 2.3, we have in \( \mathcal{L} \) every element can be expressed as linear combination of canonical words.

Denote by \( S \) the set constituted by the relations (6)–(10). Define \( \cdots > x_1^{(2)} > x_0^{(2)} > a > b > x_1 > y_n^{(1)} > y_0^{(1)} > y_1^{(2)} > \cdots \). We can see that in \( S \) there are no compositions unless for the ambiguity \( x, x, x \). But this case is trivial. Hence \( S \) is a Gröbner–Shirshov basis in \( \text{Lie}(X) \) on deg-lex ordering which implies that \( \mathcal{L} \) can be embedded into \( \mathcal{L} \). By (7)–(10), \( \mathcal{L} \) is a simple Lie algebra generated by \( \{a, b\} \).
5. Associative Differential Algebras

Composition-Diamond lemma for associative differential algebras with unit is established in a recent paper [17]. By applying this lemma in this section, we show that:

(i) Every countably generated associative differential algebra can be embedded into a two-generated associative differential algebra. (ii) Any associative differential algebra can be embedded into a one-generated associative differential algebra. (iii) Every countably generated associative differential algebra with countable set of differential operations over a countable field $k$ can be embedded into a simple two-generated associative differential algebra.

Let $A$ be an associative algebra over a field $k$ with unit. Let $D$ be a set of multiple linear operations on $A$. Then $A$ is called an associative differential algebra with differential operations on $A$ or $D$-algebra, for short, if for any $D \in D$, $a, b \in A$,

$$D(ab) = D(a)b + aD(b).$$

Let $D = \{D_j | j \in J\}$. For any $m = 0, 1, \ldots$ and $\bar{j} = (j_1, \ldots, j_m) \in J^m$, denote by $D^\bar{j} = D_{j_1}D_{j_2}\cdots D_{j_m}$ and $D^{\omega}(X) = \{D^\bar{j}(x) | x \in X, \bar{j} \in J^m, m \geq 0\}$, where $D^0(x) = x$. Let $T = (D^\omega(X))^*$ be the free monoid generated by $D^\omega(X)$. For any $u = D^{\bar{i}_1}(x_1)D^{\bar{i}_2}(x_2)\cdots D^{\bar{i}_n}(x_n) \in T$, the length of $u$, denoted by $|u|$, is defined to be $n$. In particular, $|1| = 0$.

Let $k(X; D) = kT$ be the $k$-algebra spanned by $T$. For any $D_j \in D$, we define the linear map $D_j : k(X; D) \rightarrow k(X; D)$ by induction on $|u|$ for $u \in T$:

(i) $D_j(1) = 0$.
(ii) Suppose that $u = D^\bar{i}(x) = D_{i_1}D_{i_2}\cdots D_{i_m}(x)$. Then $D_j(u) = D_jD_{i_1}D_{i_2}\cdots D_{i_m}(x)$.
(iii) Suppose that $u = D^\bar{i}(x) \cdot v, v \in T$. Then $D_j(u) = (D_jD^\bar{i}(x)) \cdot v + D^\bar{i}(x) \cdot D_j(v)$.

Then, $k(X; D)$ is a free associative differential algebra generated by $X$ with differential operators $D$ (see [17]).

Let $D = \{D_j | j \in J\}$, $X$ and $J$ well-ordered sets, $D^\bar{i}(x) = D_{i_1}D_{i_2}\cdots D_{i_m}(x) \in D^\omega(X)$. Denote by

$$wt(D^\bar{i}(x)) = (x; m, i_1, i_2, \ldots, i_m).$$

Then, we order $D^\omega(X)$ as follows:

$$D^{\bar{i}}(x) < D^{\bar{j}}(y) \iff wt(D^{\bar{i}}(x)) < wt(D^{\bar{j}}(y)) \text{ lexicographically.}$$

It is easy to check this ordering is a well ordering on $D^\omega(X)$.

Now, we order $T = (D^\omega(X))^*$ by deg-lex ordering. We will use this ordering in this section.

For convenience, for any $u \in T$, we denote $\overline{D^\bar{i}(u)}$ by $d^\bar{i}(u)$.

**Theorem 5.1.** Every countably generated associative differential algebra can be embedded into a two-generated associative differential algebra.
Proof. Suppose that $A = k(X; D | S)$ is an associative differential algebra generated by $X$ with relations $S$, where $X = \{x_i \mid i \in I\}$. By Shirshov algorithm, we can assume that with the deg-lx ordering on $(D^e(X))^*$ defined as above, $S$ is a Gröbner–Shirshov basis of the free associative differential algebra $k\langle X; D \rangle$ in the sense of the paper [17]. Let $B = k(X, a, b; D | S, aab \neq ab = x_i)$. We have that with the deg-lx ordering on $(D^e(X, a, b))^*$, $\{S, aab \neq ab = x_i, i = 1,2, \ldots\}$ is a Gröbner–Shirshov basis in the free associative differential algebra $k\langle X, a, b; D \rangle$ since there are no new compositions. By the Composition-Diamond lemma in [17], $A$ can be embedded into $B$ which is generated by $\{a, b\}$.

Theorem 5.2. Every associative differential algebra can be embedded into a simple associative differential algebra.

Proof. Let $A$ be an associative differential algebra over a field $k$ with $k$-basis $\{1\} \cup X$, where $X = \{x_i \mid i \in I\}$ and $I$ is a well-ordered set.

It is clear that $S_0 = \{x_ix_j = \{x_i, x_j\}, D(x_i) = \{D(x_i)\}, i, j \in I, D \in D\}$ where $\{D(x_i)\}$ is a linear combination of $x_j, j \in I$, is a Gröbner–Shirshov basis in the free associative differential algebra $k\langle X; D \rangle$ with the deg-lx ordering on $(D^e(X))^*$, and $A$ can be expressed as

$$A = k(X; D | x_ix_j = \{x_i, x_j\}, D(x_i) = \{D(x_i)\}, i, j \in I, D \in D).$$

Let us totally order the set of monic elements of $A$. Denote by $T$ the set of indices for the resulting totally ordered set. Consider the totally ordered set $T^2 = \{((\theta, \sigma) \mid \theta, \sigma \in T\}$ and assign $(\theta, \sigma) < (\theta', \sigma')$ if either $\theta < \theta'$ or $\theta = \theta'$ and $\sigma < \sigma'$. Then $T^2$ is also totally ordered set.

For each ordered pair of elements $f_\theta, f_\sigma \in A, \theta, \sigma \in T$, introduce the letters $x_{\theta_\sigma}, y_{\theta_\sigma}$.

Let $A_1$ be the associative differential algebra given by the generators

$$X_1 = \{x_i, y_{\theta_\sigma}, x_{\theta\tau} \mid i \in I, \theta, \sigma, \theta, \tau \in T\}$$

and the defining relations

$$S = \{x_ix_j = \{x_i, x_j\}, D(x_i) = \{D(x_i)\},$$

$$x_{\theta_\sigma}f_\theta y_{\theta_\sigma} = f_\sigma \mid i, j \in I, D \in D, (\theta, \sigma) \in T^2\}.$$

We can have that with the deg-lx ordering on $(D^e(X_1))^*$, $S$ is a Gröbner–Shirshov basis in the free associative differential algebra $k\langle X_1; D \rangle$ in the sense of the paper [17] since there are no new compositions. Thus, by the Composition-Diamond lemma in [17], $A$ can be embedded into $A_1$. The relations $x_{\theta_\sigma}f_\theta y_{\theta_\sigma} = f_\sigma$ of $A_1$ provide that in $A_1$ every monic element $f_\theta$ of the subalgebra $A$ generates an ideal containing algebra $A$.

Mimicking the construction of the associative differential algebra $A_1$ from the $A$, produce the associative differential algebra $A_2$ from $A_1$ and so on. As a result,
we acquire an ascending chain of associative differential algebras

\[ A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \]

such that every monic element \( f \in A_k \) generates an ideal in \( A_{k+1} \) containing \( A_k \).

Therefore, in the associative differential algebra

\[ A = \bigcup_{k=0}^{\infty} A_k, \]

every nonzero element generates the same ideal. Thus, \( A \) is a simple associative differential algebra.

**Theorem 5.3.** Every countably generated associative differential algebra with countable set \( D \) of differential operations over a countable field \( k \) can be embedded into a simple two-generated associative differential algebra.

**Proof.** Let \( A \) be a countably generated associative differential algebra with countable set \( D \) of differential operations over a countable field \( k \). We may assume that \( A \) has a countable \( k \)-basis \( \{1\} \cup X_0 \), where \( X_0 = \{x_i \mid i = 1, 2, \ldots\} \). Then \( A \) can be expressed as

\[ A = k\langle X_0; D \rangle \mid x_ix_j = \{x_i, x_j\}, D(x_i) = \{D(x_i)\}, i, j \in N, D \in D \rangle. \]

Let \( A_0 = k\langle X_0; D \rangle, A_0^+ = A_0 \setminus \{0\} \) and fix the bijection

\[ (A_0^+, A_0^+) \leftrightarrow \{(x_m^{(1)}, y_m^{(1)}) \mid m \in N\}. \]

Let \( X_1 = X_0 \cup \{x_m^{(1)}, y_m^{(1)}, a, b \mid m \in N\} \), \( A_1 = k\langle X_1; D \rangle, A_1^+ = A_1 \setminus \{0\} \) and fix the bijection

\[ (A_1^+, A_1^+) \leftrightarrow \{(x_m^{(2)}, y_m^{(2)}) \mid m \in N\}. \]

\[ \vdots \]

Let \( X_{n+1} = X_n \cup \{x_m^{(n+1)}, y_m^{(n+1)} \mid m \in N\}, n \geq 1, A_{n+1} = k\langle X_{n+1}; D \rangle, A_{n+1}^+ = A_{n+1} \setminus \{0\} \) and fix the bijection

\[ (A_{n+1}^+, A_{n+1}^+) \leftrightarrow \{(x_m^{(n+2)}, y_m^{(n+2)}) \mid m \in N\}. \]

\[ \vdots \]

Consider the chain of the free associative differential algebras

\[ A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots. \]

Let \( X = \bigcup_{n=0}^{\infty} X_n \). Then \( k\langle X; D \rangle = \bigcup_{n=0}^{\infty} A_n \).

Now, define the desired associative differential algebra \( A \). Take the set \( X \) as the set of the generators for this algebra and take the following relations as one part of
the relations for this algebra

\[ x_i x_j = \{x_i, x_j\}, \quad D(x_i) = \{D(x_i)\}, \quad i, j \in N, \quad D \in \mathcal{D} \quad (11) \]

\[ aa(ab)^n \bar{2}^{m+1} ab = x_i^{(n)}, \quad m, n \in N \quad (12) \]

\[ aa(ab)^n \bar{2}^m ab = y_m^{(n)}, \quad m, n \in N \quad (13) \]

\[ aabbab = x_1. \quad (14) \]

Before we introduce the another part of the relations on \( A \), let us define canonical words of the algebras \( A_n, n \geq 0 \). An element in \( (D^e(X_n))^* \) without subwords of the form \( d^e(u) \) where \( u \) is the leading terms of \( (11) \), is called a canonical word of \( A_0 \). An element in \( (D^e(X_1))^* \) without subwords of the form \( d^e(u) \) where \( u \) is the leading terms of \( (11) \)–(14) and

\[ (x_m^{(1)})^{g(0)} + 1 f(0) y_m^{(1)}, \]

where \( (x_m^{(1)}, y_m^{(1)}) \leftrightarrow (f(0), g(0)) \in (A_0^+, A_0^+) \) such that \( f(0), g(0) \) are nonzero linear combination of canonical words of \( A_0 \), is called a canonical word of \( A_1 \). Suppose that we have defined canonical word of \( A_k, k < n \). An element in \( (D^e(X_n))^* \) without subwords of the form \( d^e(u) \) where \( u \) is the leading terms of \( (11) \)–(14) and

\[ (x_m^{(1)})^{g(0)} + 1 f(0) y_m^{(1)}, \]

where \( (x_m^{(1)}, y_m^{(1)}) \leftrightarrow (f(0), g(0)) \in (A_0^+, A_0^+) \) such that \( f(0), g(0) \) are nonzero linear combination of canonical words of \( A_n \), is called a canonical word of \( A_n \).

Then the another part of the relations on \( A \) are the following:

\[ (x_m^{(n)})^{g(n)} + 1 f(n-1) y_m^{(n)} - g(n-1) = 0, \quad m, n \in N \quad (15) \]

where \( (x_m^{(n)}, y_m^{(n)}) \leftrightarrow (f(n-1), g(n-1)) \in (A_{n-1}^+, A_{n-1}^+) \) such that \( f(n-1), g(n-1) \) are nonzero linear combination of canonical words of \( A_{n-1} \).

We can get that in \( A \) every element can be expressed as linear combination of canonical words.

Denote by \( S \) the set constituted by the relations \( (11) \)–(15). We can have that with the deg-lex ordering on \( (D^e(X))^* \) defined as above, \( S \) is a Gröbner–Shirshov basis in \( k(X; \mathcal{D}) \) since in \( S \) there are no compositions except for the ambiguity \( x_i x_j x_k \) which is a trivial case. This implies that \( A \) can be embedded into \( A \). By \( (12) \)–(15), \( A \) is a simple associative differential algebra generated by \( \{a, b\} \).

6. Associative Algebras with Multiple Operations

Composition-Diamond lemma for associative algebra with multiple operations \( \Omega \) (associative \( \Omega \)-algebra, for short) is established in a recent paper [12]. By applying this lemma, we show in this section that: (i) Every countably generated associative \( \Omega \)-algebra can be embedded into a two-generated associative \( \Omega \)-algebra. (ii) Any associative \( \Omega \)-algebra can be embedded into a simple associative \( \Omega \)-algebras. (iii) Each countably generated associative \( \Omega \)-algebra with countable
multiple operations $\Omega$ over a countable field $k$ can be embedded into a simple two-generated associative $\Omega$-algebra.

The concept of multi-operations algebras ($\Omega$-algebras) was first introduced by A. G. Kurosh in [30, 31].

Let $k$ be a field. An associative algebra with multiple linear operations is an associative $k$-algebra $A$ with a set $\Omega$ of multi-linear operations.

Let $X$ be a set and

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n,$$

where $\Omega_n$ is the set of $n$-ary operations, for example, $\text{ary}(\delta) = n$ if $\delta \in \Omega_n$.

Denote by $S(X)$ the free semigroup without identity generated by $X$.

For any non-empty set $Y$ (not necessarily a subset of $S(X)$), let

$$\Omega(Y) = \bigcup_{n=1}^{\infty} \{ \delta(x_1, x_2, \ldots, x_n) \mid \delta \in \Omega_n, x_i \in Y, i = 1, 2, \ldots, n \}.$$

Define

$$\mathcal{S}_0 = S(X),$$
$$\mathcal{S}_1 = S(X \cup \Omega(\mathcal{S}_0)),$$
$$\vdots$$
$$\mathcal{S}_n = S(X \cup \Omega(\mathcal{S}_{n-1})),$$
$$\vdots$$

Then we have

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_n \subset \cdots.$$

Let

$$\mathcal{S}(X) = \bigcup_{n \geq 0} \mathcal{S}_n.$$

Then, we can see that $\mathcal{S}(X)$ is a semigroup such that $\Omega(\mathcal{S}(X)) \subseteq \mathcal{S}(X)$.

For any $u \in \mathcal{S}(X)$, $\text{dep}(u) = \min\{n \mid u \in \mathcal{S}_n\}$ is called the depth of $u$.

Let $k(X; \Omega)$ be the $k$-algebra spanned by $\mathcal{S}(X)$. Then, the element in $\mathcal{S}(X)$ (respectively, $k(X; \Omega)$) is called a $\Omega$-word (respectively, $\Omega$-polynomial).

Extend linearly each map $\delta \in \Omega_n$,

$$\delta : \mathcal{S}(X)^n \to \mathcal{S}(X), \quad (x_1, x_2, \ldots, x_n) \mapsto \delta(x_1, x_2, \ldots, x_n)$$

to $k(X; \Omega)$. Then, $k(X; \Omega)$ is a free associative algebra with multiple linear operators $\Omega$ on the set $X$ (see [12]).
Let $X$ and $\Omega$ be well-ordered sets. We order $X^*$ by the deg-lex ordering. For any $u \in \mathfrak{S}(X)$, $u$ can be uniquely expressed without brackets as

$$u = u_0 \delta_i, x_{i_1} u_1 \cdots \delta_i, x_{i_k} u_t,$$

where each $u_i \in X^*$, $\delta_i \in \Omega$, $x_{i_k} = (x_{k_1}, x_{k_2}, \ldots, x_{k_{i_k}}) \in \mathfrak{S}(X)^{i_k}$.

Denote by

$$wt(u) = (t, \delta_1, x_{i_1}, \ldots, \delta_i, x_{i_k}, u_0, u_1, \ldots, u_t).$$

Then, we order $\mathfrak{S}(X)$ as follows: for any $u, v \in \mathfrak{S}(X)$,

$$u > v \iff wt(u) > wt(v)$$

lexicographically (16).

By induction on $\text{dep}(u) + \text{dep}(v)$.

It is clear that the ordering (16) is a monomial ordering on $\mathfrak{S}(X)$ (see [12]).

Denote by $\text{deg}_\Omega(u)$ the number of $\delta$ in $u$ where $\delta \in \Omega$, for example, if $u = x_1 \delta_1(x_2) \delta_3(x_2, x_1, \delta_1(x_3))$, then $\text{deg}_\Omega(u) = 3$.

Theorem 6.1. Every countably generated associative $\Omega$-algebra can be embedded into a two-generated associative $\Omega$-algebra.

Proof. Suppose that $A = k\langle X; \Omega \mid S \rangle$ is an associative $\Omega$-algebra generated by $X$ with relations $S$, where $X = \{x_i, i = 1, 2, \ldots\}$. By Shirshov algorithm, we can assume that $S$ is a Gröbner–Shirshov basis of the free associative $\Omega$-algebra $k\langle X; \Omega \rangle$ in the sense of the paper [12] with the ordering (16). Let $H = k\langle X, a, b; \Omega \mid S, aab'ab = x_i, i = 1, 2, \ldots \rangle$. We can check that $\{S, aab'ab = x_i, i = 1, 2, \ldots \}$ is a Gröbner–Shirshov basis in the free associative $\Omega$-algebra $k\langle X, a, b; \Omega \rangle$ since there are no new compositions. By the Composition-Diamond lemma in [12], $A$ can be embedded into $H$ which is generated by $(a, b)$. □

Theorem 6.2. Every associative $\Omega$-algebra can be embedded into a simple associative $\Omega$-algebra.

Proof. Let $A$ be an associative $\Omega$-algebra over a field $k$ with $k$-basis $X = \{x_i | i \in I\}$ where $I$ is a well ordered set. Denote by

$$S = \{x_i x_j = \{x_i, j \}, \delta_n(x_{k_1}, \ldots, x_{k_n}) = \{\delta_n(x_{k_1}, \ldots, x_{k_n}) \}, i, j, k_1, \ldots, k_n \in I, \delta_n \in \Omega, n \in N\},$$

where $\{\delta_n(x_{k_1}, \ldots, x_{k_n}) \}$ is a linear combination of $x_i, i \in I$. Then in the sense of the paper [12], $S$ is a Gröbner–Shirshov basis in the free associative $\Omega$-algebra $k\langle X; \Omega \rangle$ with the ordering (16). Therefore $A$ can be expressed as

$$A = k\langle X; \Omega \mid S \rangle.$$

Let us totally order the set of monic elements of $A$. Denote by $T$ the set of indices for the resulting totally ordered set. Consider the totally ordered set $T^2 = \{(\theta, \sigma)\}$
and assign \((\theta, \sigma) < (\theta', \sigma')\) if either \(\theta < \theta'\) or \(\theta = \theta'\) and \(\sigma < \sigma'\). Then \(T^2\) is also totally ordered set.

For each ordered pair of elements \(f_\theta, f_\sigma \in A, \theta, \sigma \in T\), introduce the letters \(x_{\theta\sigma}, y_{\theta\sigma}\).

Let \(A_1\) be the associative \(\Omega\)-algebra given by the generators

\[ X_1 = \{x_i, y_{\theta\sigma}, x_{\theta\sigma} \mid i \in I, \theta, \sigma, \tau \in T\} \]

and the defining relations \(S_1\) where \(S_1\) is the union of \(S\) and

\[ x_{\theta\sigma} f_\theta y_{\theta\sigma} = f_\sigma, \quad (\theta, \sigma) \in T^2. \]

We can have that in \(S_1\) there are no compositions unless for the ambiguity \(x_i x_j x_k\). But this case is trivial. Hence \(S_1\) is a Gröbner–Shirshov basis of the free associative \(\Omega\)-algebra \(k\langle X_1; \Omega \rangle\) in the sense of the paper [12] with the ordering \((16)\). Thus, by the Composition-Diamond lemma in \([12]\), \(A\) can be embedded into \(A_1\). The relations \(x_{\theta\sigma} f_\theta y_{\theta\sigma} = f_\sigma\) of \(A_1\) provide that in \(A_1\) every monic element \(f_\theta\) of the subalgebra \(A\) generates an ideal containing algebra \(A\).

Mimicking the construction of the associative \(\Omega\)-algebra \(A_1\) from the \(A\), produce the associative \(\Omega\)-algebras \(A_2, A_3, \ldots\) from \(A_1\) and so on. As a result, we acquire an ascending chain of associative \(\Omega\)-algebras \(A = A_0 \subset A_1 \subset A_2 \subset \cdots\) such that every nonzero element generates the same ideal. Let \(A = \bigcup_{k=0}^{\infty} A_k\). Then \(A\) is a simple associative \(\Omega\)-algebra.

**Theorem 6.3.** Every countably generated associative \(\Omega\)-algebra with countable multiple operations \(\Omega\) over a countable field \(k\) can be embedded into a simple two-generated associative \(\Omega\)-algebra.

**Proof.** Let \(A\) be a countably generated associative \(\Omega\)-algebra with countable multiple operations \(\Omega\) over a countable field \(k\). We may assume that \(A\) has a countable \(k\)-basis \(X_0 = \{x_i \mid i = 1, 2, \ldots\}\). Denote by

\[ S = \{x_i x_j = \{x_i, x_j\}, \delta_n(x_{k_1}, \ldots, x_{k_n}) = \{\delta_n(x_{k_1}, \ldots, x_{k_n})\} \mid i, j, k_1, \ldots, k_n \in N, \delta_n \in \Omega_n, n \in N\}. \]

Then \(A\) can be expressed as \(A = k(\langle X_0; \Omega \mid S \rangle\).

Let \(A_0 = k\langle X_0; \Omega \rangle, A_0^+ = A_0 \setminus \{0\}\) and fix the bijection

\[ (A_0^+, A_0^+) \leftrightarrow \{(x_m^{(1)}, y_m^{(1)}) \mid m \in N\}. \]

Let \(X_1 = X_0 \cup \{x_m^{(1)}, y_m^{(1)}, a, b \mid m \in N\}\), \(A_1 = k\langle X_1; \Omega \rangle, A_1^+ = A_1 \setminus \{0\}\) and fix the bijection

\[ (A_1^+, A_1^+) \leftrightarrow \{(x_m^{(2)}, y_m^{(2)}) \mid m \in N\}. \]

\vdots
Let \( X_{n+1} = X_n \cup \{x_m^{(n+1)}, y_m^{(n+1)} | m \in N \} \), \( n \geq 1, A_{n+1} = k\langle X_{n+1}; \Omega \rangle, A_{n+1}^{+} = A_{n+1} \setminus \{0\} \) and fix the bijection
\[
(A_{n+1}^{+}, A_{n+1}^{+}) \leftrightarrow \{(x_m^{(n+2)}, y_m^{(n+2)}), m \in N\}.
\]

Consider the chain of the free associative \( \Omega \)-algebras
\[
A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots.
\]

Let \( X = \bigcup_{n=0}^{\infty} X_n \). Then \( k\langle X; \Omega \rangle = \bigcup_{n=0}^{\infty} A_n \).

Now, define the desired algebra \( A \). Take the set \( X \) as the set of the generators for this algebra and take the union of \( S \) and the following relations as one part of the relations for this algebra
\[
aa(ab)^n b^{2m+1}ab = x_m^{(n)}, \quad m, n \in N \quad (17)
\]
\[
aa(ab)^n b^{2m}ab = y_m^{(n)}, \quad m, n \in N \quad (18)
\]
\[
aabbab = x_1. \quad (19)
\]

Before we introduce the another part of the relations on \( A \), let us define canonical words of the algebras \( A_n, n \geq 0 \). A \( \Omega \)-word in \( X_0 \) without subwords that are the leading terms of \( s (s \in S) \) is called a canonical word of \( A_0 \). A \( \Omega \)-word in \( X_1 \) without subwords that are the leading terms of \( s (s \in S \cup \{(17)-(19)\}) \) and without subwords of the form
\[
(\delta_1(x_m^{(1)})^{\deg f^{(0)}})^{y_m^{(1)}} y_m^{(1)},
\]
where \( (x_m^{(1)}, y_m^{(1)}) \leftrightarrow (f^{(0)}, g^{(0)}) \in (A_0^+, A_0^+) \) such that \( f^{(0)}, g^{(0)} \) are nonzero linear combination of canonical words of \( A_0 \), is called a canonical word of \( A_1 \). Suppose that we have defined canonical word of \( A_k, k < n \). A \( \Omega \)-word in \( X_n \) without subwords that are the leading terms of \( s (s \in S \cup \{(17)-(19)\}) \) and without subwords of the form
\[
(\delta_1(x_m^{(k+1)})^{\deg f^{(k)}})^{y_m^{(k+1)}} y_m^{(k+1)},
\]
where \( (x_m^{(k+1)}, y_m^{(k+1)}) \leftrightarrow (f^{(k)}, g^{(k)}) \in (A_k^+, A_k^+) \) such that \( f^{(k)}, g^{(k)} \) are nonzero linear combination of canonical words of \( A_k \), is called a canonical word of \( A_n \).

Then the another part of the relations on \( A \) are the following:
\[
(\delta_1(x_m^{(n)})^{\deg f^{(n-1)}})^{f^{(n-1)}} y_m^{(n)} - g^{(n-1)} = 0, \quad m, n \in N \quad (20)
\]
where \( (x_m^{(n)}, y_m^{(n)}) \leftrightarrow (f^{(n-1)}, g^{(n-1)}) \in (A_{n-1}^+, A_{n-1}^+) \) such that \( f^{(n-1)}, g^{(n-1)} \) are nonzero linear combination of canonical words of \( A_{n-1} \).

We can see that in \( A \) every element can be expressed as linear combination of canonical words.

Denote by \( S_1 = S \cup \{(17)-(20)\} \). We can have that with the ordering \( (16), S_1 \) is a Gröbner–Shirshov basis in \( k\langle X; \Omega \rangle \) in the sense of the paper [12] since in \( S_1 \) there
are no compositions except for the ambiguity $x_i x_j x_k$ which is a trivial case. This implies that $A$ can be embedded into $A$. By (17)–(20), $A$ is a simple associative $\Omega$-algebra generated by \{a, b\}.

7. Associative $\lambda$-Differential Algebras

In this section, by applying the Composition-Diamond lemma for associative $\Omega$-algebras in [12], we show that: (i) Each countably generated associative $\lambda$-differential algebra can be embedded into a two-generated associative $\lambda$-differential algebra. (ii) Each associative $\lambda$-differential algebra can be embedded into a simple associative $\lambda$-differential algebra. (iii) Each countably generated associative $\lambda$-differential algebra over a countable field $k$ can be embedded into a simple two-generated associative $\lambda$-differential algebra.

Let $k$ be a commutative ring with unit and $\lambda \in k$. An associative $\lambda$-differential algebra over $k$ ([23]) is an associative $k$-algebra $R$ together with a $k$-linear operator $D : R \to R$ such that

$$D(xy) = D(x)y +xD(y) + \lambda D(x)D(y), \quad \forall \ x, y \in R.$$  

Any associative $\lambda$-differential algebra is also an associative algebra with one operator $\Omega = \{D\}$.

In this section, we will use the notations given in Sec. 6.

Let $X$ be well-ordered and $k\langle X; D \rangle$ the free associative algebra with one operator $\Omega = \{D\}$ defined in Sec. 6.

For any $u \in \mathcal{S}(X)$, $u$ has a unique expression

$$u = u_1 u_2 \cdots u_n,$$

where each $u_i \in X \cup D(\mathcal{S}(X))$. Denote by $\deg_x(u)$ the number of $x \in X$ in $u$, for example, if $u = D(x_1 x_2)D(D(x_1))x_3 \in \mathcal{S}(X)$, then $\deg_x(u) = 4$. Let

$$wt(u) = (\deg_x(u), u_1, u_2, \ldots, u_n).$$

Now, we order $\mathcal{S}(X)$ as follows: for any $u, v \in \mathcal{S}(X)$,

$$u > v \iff wt(u) > wt(v) \text{ lexicographically}$$  \hspace{1cm} (21)

where for each $t$, $u_t > v_t$ if one of the following holds:

(a) $u_t, v_t \in X$ and $u_t > v_t$;
(b) $u_t = D(u'_t), v_t \in X$;
(c) $u_t = D(u'_t), v_t = D(v'_t)$ and $u'_t > v'_t$.

Then the ordering (21) is a monomial ordering on $\mathcal{S}(X)$ (see [12]).

Lemma 7.1 ([12], Theorem 5.1). With the ordering (21) on $\mathcal{S}(X)$,

$$S_0 = \{D(uv) - D(u)v - uD(v) - \lambda D(u)D(v) \mid u, v \in \mathcal{S}(X)\}$$

is a Gröbner–Shirshov basis in the free $\Omega$-algebra $k\langle X; D \rangle$ where $\Omega = \{D\}$.
Lemma 7.2. Let \( A \) be an associative \( \lambda \)-differential algebra with \( k \)-basis \( X = \{ x_i \mid i \in I \} \). Then \( A \) has a representation \( A = k(X; D \mid S) \), where \( S = \{ x_i x_j = \{ x_i, x_j \}, D(x_i) = \{ D(x_i) \}, D(x_i x_j) = D(x_i) x_j + x_i D(x_j) + \lambda D(x_i) D(x_j) \mid i, j \in I \} \).

Moreover, if \( I \) is a well-ordered set, then with the ordering (21) on \( \mathfrak{S}(X) \), \( S \) is a Gröbner–Shirshov basis in the free \( \Omega \)-algebra \( k(X; D) \) in the sense of [12].

Proof. Clearly, \( k(X; D \mid S) \) is an associative \( \lambda \)-differential algebra. By the Composition-Diamond lemma in [12], it suffices to check that with the ordering (21) on \( \mathfrak{S}(X) \), \( S \) is a Gröbner–Shirshov basis in \( k(X; D) \) in the sense of [12].

The ambiguities \( w \) of all possible compositions of \( \Omega \)-polynomials in \( S \) are:

1) \( x_i x_j x_k, i, j, k \in I \),
2) \( D(x_i x_j), i, j \in I \).

We will check that each composition in \( S \) is trivial \( \text{mod}(S, w) \).

For 1), the result is trivial.

For 2), let \( f = D(x_i x_j) - D(x_i) x_j - x_i D(x_j) - \lambda D(x_i) D(x_j), g = x_i x_j - \{ x_i, x_j \}, i, j \in I \). Then \( w = D(x_i x_j) \) and

\[
(f, g)w = -D(x_i) x_j - x_i D(x_j) - \lambda D(x_i) D(x_j) + D(\{ x_i, x_j \})
\equiv \{ D(\{ x_i, x_j \}) \} - \{ \{ D(x_i) \}, x_j \} - \{ x_i, \{ D(x_j) \} \} - \lambda \{ \{ D(x_i) \}, \{ D(x_j) \} \}
\equiv 0 \text{ mod}(S, w).
\]

This shows that \( S \) is a Gröbner–Shirshov basis in the free \( \Omega \)-algebra \( k(X; D) \). □

Now we get the embedding theorems for associative \( \lambda \)-differential algebras.

Theorem 7.3. Every countably generated associative \( \lambda \)-differential algebra over a field can be embedded into a two-generated associative \( \lambda \)-differential algebra.

Proof. Let \( A \) be a countably generated associative \( \lambda \)-differential algebra over a field \( k \). We may assume that \( A \) has a countable \( k \)-basis \( X = \{ x_i \mid i = 1, 2, \ldots \} \). By Lemma 7.2, \( A = k(X; D \mid S) \), where \( S = \{ x_i x_j = \{ x_i, x_j \}, D(x_i) = \{ D(x_i) \}, D(x_i x_j) = D(x_i) x_j + x_i D(x_j) + \lambda D(x_i) D(x_j) \mid i, j \in N \} \).

Let \( H = k\langle X, a, b; D \mid S_1 \rangle \) where

\[
S_1 = \{ x_i x_j = \{ x_i, x_j \}, D(x_i) = \{ D(x_i) \}, D(uv) = D(u)v + uD(v) + \lambda D(u)D(v),
aab^*b = x_i \mid u, v \in \mathfrak{S}(X, a, b), i, j \in N \}.
\]

We want to prove that \( S_1 \) is also a Gröbner–Shirshov basis in the free \( \Omega \)-algebra \( k(X, a; D) \) with the ordering (21). Now, let us check all the possible compositions...
in $S_1$. The ambiguities $w$ of all possible compositions of $\Omega$-polynomials in $S_1$ are:

1) $x_i x_j x_k$
2) $D(u|_{x_i x_j} v)$
3) $D(u|_{x_i x_j} v)$
4) $D(u|_{D(x_i)} v)$
5) $D(u|_{D(x_j)} v)$
6) $D(u|_{D(u v)} v)$
7) $D(u|_{D(u v)} v)$
8) $D(u|_{a a b \cdot a b v})$
9) $D(u|_{a a b \cdot a b})$

where $u, v, u_1, v_1 \in S(X, a, b), x_i, x_j, x_k \in X$. We have to check that all these compositions are trivial mod($S_1, w$). In fact, by Lemma 7.1 and since $S$ is a Gröbner–Shirshov basis in $k(X; D)$, we need only to check 2)-5), 8), 9). Here, for example, we just check 3), 4), 8). Others are similarly proved.

For 3), let $f = D(u|_{x_i x_j}) - D(u|_{x_i x_j}) - u D(v|_{x_i x_j}) - \lambda D(u) D(v|_{x_i x_j})$, $g = x_i x_j - \{ x_i, x_j \}$, $u, v \in S(X, a, b), x_i, x_j \in X$. Then $w = D(u|_{x_i x_j}) v$ and

$$
(f, g)_w = -D(u|_{x_i x_j}) v - u D(v|_{x_i x_j}) - \lambda D(u) D(v|_{x_i x_j}) + D(u|_{x_i x_j}) v
$$

$\equiv -D(u|_{x_i x_j}) v - u D(v|_{x_i x_j}) - \lambda D(u) D(v|_{x_i x_j}) + D(u|_{x_i x_j}) v
\equiv 0.$

For 4), let $f = D(u|_{D(x_j)} v) - D(u|_{D(x_j)} v) - u|_{D(x_j)} D(v) - \lambda D(u|_{D(x_j)}) D(v)$, $g = D(x_i) - \{ D(x_i) \}$, $u, v, D(x_i) \in S(X, a, b), x_i \in X$. Then $w = D(u|_{D(x_i)}) v$ and

$$(f, g)_w = -D(u|_{D(x_i)}) v - u|_{D(x_i)} D(v) - \lambda D(u|_{D(x_i)}) D(v) + D(u|_{D(x_i)}) v
\equiv -D(u|_{D(x_i)}) v - u|_{D(x_i)} D(v) - \lambda D(u|_{D(x_i)}) D(v) + D(u|_{D(x_i)}) v
\equiv 0.$$

For 8), let $f = D(u|_{a a b \cdot a b v}) - D(u|_{a a b \cdot a b v}) - u|_{a a b \cdot a b} D(v) - \lambda D(u|_{a a b \cdot a b}) D(v)$, $g = a a b \cdot a b - x_i$, $u, v \in S(X, a, b), x_i \in X$. Then $w = D(u|_{a a b \cdot a b}) v$ and

$$
(f, g)_w = -D(u|_{a a b \cdot a b}) v - u|_{a a b \cdot a b} D(v) - \lambda D(u|_{a a b \cdot a b}) D(v) + D(u|_{a a b \cdot a b}) v
\equiv D(u|_{a a b \cdot a b}) v - u|_{a a b \cdot a b} D(v) - \lambda D(u|_{a a b \cdot a b}) D(v)
\equiv 0.$$

So $S_1$ is a Gröbner–Shirshov basis in $k(X, a, b; D)$. By the Composition-Diamond lemma in [12], $A$ can be embedded into $H$ which is generated by $\{ a, b \}$.

**Theorem 7.4.** Every associative $\lambda$-differential algebra over a field can be embedded into a simple associative $\lambda$-differential algebra.

**Proof.** Let $A$ be an associative $\lambda$-differential algebra over a field $k$ with basis $X = \{ x_i | i \in I \}$ where $I$ is a well-ordered set. Then by Lemma 7.2, $A$ can be expressed as $A = k(X; D| S)$ where $S = \{ x_i, x_j = \{ x_i, x_j \}, D(x_i) = \{ D(x_i) \}, D(x_i x_j) = D(x_i) x_j + x_i D(x_j) + \lambda D(x_i) D(x_j) \}$ for $i, j \in I$ and $S$ is a Gröbner–Shirshov basis in $k(X; D)$ with the ordering (21). Let us totally order the set of monic elements of $A$. Denote by $T$ the set of indices for the resulting totally ordered set. Consider
the totally ordered set \( T^2 = \{(\theta, \sigma)\} \) and assign \((\theta, \sigma) < (\theta', \sigma')\) if either \(\theta < \theta'\) or \(\theta = \theta'\) and \(\sigma < \sigma'\). Then \( T^2 \) is also totally ordered set.

For each ordered pair of elements \( f_\theta, f_\sigma \in A, \theta, \sigma \in T \), introduce the letters \( x_{\theta \sigma}, y_{\theta \sigma} \).

Let \( A_1 \) be the associative \( \lambda \)-differential algebra given by the generators

\[
X_1 = \{x_i, y_{\theta \sigma}, x_{\tau \sigma} \mid i \in I, \theta, \sigma, \theta, \tau \in T\}
\]

and the defining relations

\[
x_i x_j = \{x_i, x_j\}, \quad i, j \in I,
\]

\[
D(x_i) = \{D(x_i)\}, \quad i \in I,
\]

\[
D(uv) = D(u)v + uD(v) + \lambda D(u)D(v), \quad u, v \in \mathcal{S}(X_1),
\]

\[
x_{\theta \sigma} f_\theta y_{\theta \sigma} = f_\sigma, \quad (\theta, \sigma) \in T^2.
\]

We want to prove that these relations is also a Gröbner–Shirshov basis in \( k\langle X_1; D \rangle \) with the same ordering (21). Now, let us check all the possible compositions. The ambiguities \( w \) of all possible compositions of \( \Omega \)-polynomials are:

1) \( x_i x_j x_k \)  
2) \( D(u|x,x,v) \)  
3) \( D(u|v|x,x,j) \)  
4) \( D(u|D(x_i)v) \)  
5) \( D(u|D(x_i)|v) \)  
6) \( D(u|D(x_i|v)) \)  
7) \( D(u|D(u_1|v_1)) \)  
8) \( D(u|x_{\theta \sigma} f_\theta y_{\theta \sigma}) \)  
9) \( D(u|x_{\theta \sigma} f_\theta y_{\theta \sigma}) \)

where \( u, v, u_1, v_1 \in \mathcal{S}(X_1), x_i, x_j, x_k \in X, (\theta, \sigma) \in T^2 \).

In fact, by Lemma 7.1 and since \( S \) is a Gröbner–Shirshov basis in \( K\langle X; D \rangle \), we just need to check 2)–5), 8), 9). Here, for example, we just check 8). Others are similarly proved.

Let \( f = D(u|x_{\theta \sigma} f_\theta y_{\theta \sigma}) \) and \( g = x_{\theta \sigma} f_\theta y_{\theta \sigma} - f_\sigma = x_{\theta \sigma} f_\theta y_{\theta \sigma} + x_{\theta \sigma} f_\theta y_{\theta \sigma} - f_\sigma \), where \( f_\theta = f_\theta + f_\theta, u, v \in \mathcal{S}(X_1), (\theta, \sigma) \in T^2 \). Then \( w = D(u|x_{\theta \sigma} f_\theta y_{\theta \sigma}) \) and

\[
(f,g)_w = -D(u|x_{\theta \sigma} f_\theta y_{\theta \sigma})v - u|x_{\theta \sigma} f_\theta y_{\theta \sigma} D(v) - \lambda D(u|x_{\theta \sigma} f_\theta y_{\theta \sigma}) D(v)
\]

\[
+ D(u|(-x_{\theta \sigma} f_\theta y_{\theta \sigma} + f_\sigma)v)
\]

\[
\equiv D(u|(-x_{\theta \sigma} f_\theta y_{\theta \sigma} + f_\sigma)v) - D(u|(-x_{\theta \sigma} f_\theta y_{\theta \sigma} + f_\sigma)v) - u|(-x_{\theta \sigma} f_\theta y_{\theta \sigma} + f_\sigma)v)
\]

\[
- \lambda D(u|(-x_{\theta \sigma} f_\theta y_{\theta \sigma} + f_\sigma)v)
\]

\[
\equiv 0.
\]

Thus, by the Composition-Diamond lemma in [12], \( A \) can be embedded into \( A_1 \). The relations \( x_{\theta \sigma} f_\theta y_{\theta \sigma} = f_\sigma \) of \( A_1 \) provide that in \( A_1 \) every monic element \( f_\theta \) of the subalgebra \( A \) generates an ideal containing algebra \( A \).

Mimicking the construction of the associative \( \lambda \)-differential algebra \( A_1 \) from the \( A \), produce the associative \( \lambda \)-differential algebra \( A_2 \) from \( A_1 \) and so on. As a result, we acquire an ascending chain of associative \( \lambda \)-differential algebras
Let $\lambda$ be a countable field $k$. Every countably generated associative $\lambda$-differential algebra, and take the following relations as one part of the relations for this algebra and take the following relations as one part of the relations for this algebra.

**Theorem 7.5.** Every countably generated associative $\lambda$-differential algebra over a countable field $k$ can be embedded into a simple two-generated associative $\lambda$-differential algebra.

**Proof.** Let $A$ be a countably generated associative $\lambda$-differential algebra over a countable field $k$. We may assume that $A$ has a countable $k$-basis $X_0 = \{x_i | i = 1, 2, \ldots\}$ and can be expressed as, by Lemma 7.2, $A = k\langle X_0; D \mid S_0 \rangle$ where $S_0 = \{x_i x_j = \{x_i, x_j \}, D(x_i) = \{D(x_i)\}, D(x_i x_j) = D(x_i)x_j + x_i D(x_j) + \lambda D(x_i)D(x_j) | i, j \in N\}$ and $S_0$ is a Gröbner–Shirshov basis in $k\langle X_0; D \rangle$ with the ordering (21).

Let $A_0 = k\langle X_0; D \rangle$, $A_0^+ = A_0 \setminus \{0\}$ and fix the bijection

$$(A_0^+, A_0^+) \leftrightarrow \{(x_m^{(1)}, y_m^{(1)}), m \in N\}.$$  

Let $X_1 = X_0 \cup \{x_m^{(1)}, y_m^{(1)} | a, b | m \in N\}$, $A_1 = k\langle X_1; D \rangle$, $A_1^+ = A_1 \setminus \{0\}$ and fix the bijection

$$(A_1^+, A_1^+) \leftrightarrow \{(x_m^{(2)}, y_m^{(2)}), m \in N\}.$$  

$\vdots$

Let $X_{n+1} = X_n \cup \{x_m^{(n+1)}, y_m^{(n+1)} | m \in N\}$, $n \geq 1$, $A_{n+1} = k\langle X_{n+1}; D \rangle$, $A_{n+1}^+ = A_{n+1} \setminus \{0\}$ and fix the bijection

$$(A_{n+1}^+, A_{n+1}^+) \leftrightarrow \{(x_m^{(n+2)}, y_m^{(n+2)}), m \in N\}.$$  

$\vdots$

Consider the chain of the free $\Omega$-algebras

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots.$$  

Let $X = \bigcup_{n=0}^{\infty} X_n$. Then $k\langle X; D \rangle = \bigcup_{n=0}^{\infty} A_n$.

Now, define the desired algebra $A$. Take the set $X$ as the set of the generators for this algebra and take the following relations as one part of the relations for this algebra

$$x_i x_j = \{x_i, x_j \}, \quad D(x_i) = \{D(x_i)\}, \quad i, j \in N \quad (22)$$

$$D(uv) = D(u)v + uD(v) + \lambda D(u)D(v), \quad u, v \in \mathfrak{S}(X) \quad (23)$$

$$aa(ab)^n b^{2m+1} ab = \begin{cases} x_m^{(n)}, & m, n \in N \quad (24) \\
\end{cases}$$

$$aa(ab)^n b^{2m} ab = \begin{cases} y_m^{(n)}, & m, n \in N \quad (25) \\
\end{cases}$$

$$aabbab = x_1 \quad (26)$$
Before we introduce the another part of the relations on \( \mathcal{A} \), let us define canonical words of the algebras \( A_n \), \( n \geq 0 \). A \( \Omega \)-word in \( X_0 \) without subwords that are the leading terms of (22)–(26) and without subwords of \( A \) without subwords that are the leading terms of (22)–(26) and without subwords of the form
\[
(x_m^{(1)})^{\deg_X(g^{(0)})} f^{(0)} y_m^{(1)}
\]
where \((x_m^{(1)}, y_m^{(1)}) \leftrightarrow (f^{(0)}, g^{(0)}) \in (A_0^+, A_0^-)\) such that \( f^{(0)}, g^{(0)} \) are nonzero linear combination of canonical words of \( A_0 \), is called a canonical word of \( A_k \), \( k < n \). A \( \Omega \)-word in \( X_n \) without subwords that are the leading terms of (22)–(26) and without subwords of the form
\[
(x_m^{(k+1)})^{\deg_X(g^{(k)})} f^{(k)} y_m^{(k+1)}
\]
where \((x_m^{(k+1)}, y_m^{(k+1)}) \leftrightarrow (f^{(k)}, g^{(k)}) \in (A_k^+, A_k^-)\) such that \( f^{(k)}, g^{(k)} \) are nonzero linear combination of canonical words of \( A_k \), is called a canonical word of \( A_n \).

Then the another part of the relations on \( \mathcal{A} \) are the following:
\[
(x_m^{(n)})^{\deg_X(g^{(n-1)})} f^{(n-1)} y_m^{(n)} - g^{(n-1)} = 0, \quad m, n \in N \tag{27}
\]
where \((x_m^{(n)}, y_m^{(n)}) \leftrightarrow (f^{(n-1)}, g^{(n-1)}) \in (A_{n-1}^+, A_{n-1}^-)\) such that \( f^{(n-1)}, g^{(n-1)} \) are nonzero linear combination of canonical words of \( A_{n-1} \).

We can get that in \( \mathcal{A} \) every element can be expressed as linear combination of canonical words.

Denote by \( S \) the set constituted by the relations (22)–(27). We want to prove that \( S \) is also a Gröbner–Shirshov basis in the free \( \Omega \)-algebra \( k\langle X; D \rangle \) with the ordering (21). The ambiguities \( w \) of all possible compositions of \( \Omega \)-polynomials in \( S \) are:

1) \( x_ix_jx_k \)  2) \( D(D(u|x_ix_j)v) \)
3) \( D(u|x_ix_j) \)  4) \( D(u|D(x_i)v) \)
5) \( D(u|D(x_i)) \)  6) \( D(u|D(u_1v_1)) \)
7) \( D(u|D(u_1v_1)v) \)  8) \( D(u|_{aa(ab)^n b^{2m+1} a b} v) \)
9) \( D(u|_{aa(ab)^n b^{2m+1} ab} ) \)  10) \( D(u|_{aa(ab)^n b^{2m+1} ab} v) \)
11) \( D(u|_{aa(ab)^n b^{2m+1} ab} ) \)  12) \( D(u|_{aaabb} ) \)
13) \( D(u|_{aaabb} ) \)  14) \( D(u|_{aaabb} ) \)
15) \( D(u|_{aaabb} ) \)

where \( u, v, u_1, v_1 \in \Theta(X_1), x_i, x_j, x_k \in X \). The proof of all possible compositions to be trivial mod(\( S, w \)) is similar to that of Theorems 7.3 and 7.4. Here we omit the details. So \( S \) is a Gröbner–Shirshov basis in \( k\langle X; D \rangle \) with the ordering (21), which implies that \( \mathcal{A} \) can be embedded into \( \mathcal{A} \). By (24)–(27), \( \mathcal{A} \) is a simple associative \( \lambda \)-differential algebra generated by \( \{a, b\} \). \( \Box \)
Remark. All previous embeddings into 2-generated algebras (groups, semigroups, etc) are without distortion in the sense of Bahturin-Olshanskii [2].

References
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