Generalized Drazin invertibility of combinations of idempotents

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ABSTRACT

The paper serves as a correction to J. Math. Anal. Appl. 359 (2009) 731–738 dealing with the Drazin invertibility of combinations of idempotents \( p, q \) in a Banach algebra. As the arguments employed to do this work equally well for both the usual and generalized Drazin inverse, the latter is included in the discussion which covers the equivalence of the Drazin invertibility of \( 1 - pq, p - q, p + q \) and in general of \( \alpha p + \beta q \) with \( \alpha \beta \neq 0 \), as well as the equivalence of the Drazin invertibility of the commutator \( pq - qp \) and anticommutator \( pq + qp \) of \( p, q \).

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1. Introduction

First we briefly recall some relevant facts about the generalized Drazin inverse of Banach algebra elements (see [9]). By \( \mathcal{A} \) we denote a Banach algebra with unit 1. The spectrum of \( a \in \mathcal{A} \) is denoted by \( \sigma(a) \). The symbols \( \mathcal{A}^{\text{inv}}, \mathcal{A}^{\text{idem}}, \mathcal{A}^{\text{nil}} \) and \( \mathcal{A}^{\text{qnil}} \) denote the set of all invertible, idempotent, nilpotent and quasinilpotent elements of \( \mathcal{A} \), respectively. (An element \( a \) is \text{quasinilpotent} if \( \sigma(a) = \{0\} \).

The \text{generalized Drazin inverse} of \( a \in \mathcal{A} \) is defined as an element \( x \in \mathcal{A} \) satisfying

\[ ax = xa, \quad ax^2 = x, \quad a^2 x - a \in \mathcal{A}^{\text{qnil}}. \]  

(1.1)

Such element is unique if it exists, double commutes with \( a \), and will be denoted by \( a^D \). The set of all generalized Drazin invertible elements will be denoted by \( \mathcal{A}^D \). If \( a \in \mathcal{A}^D \) satisfies \( a^2 a^D - a \in \mathcal{A}^{\text{nil}} \), it is
called Drazin invertible, and its generalized Drazin inverse is called the Drazin inverse and is denoted by 
\( a^d \). The least nilpotency exponent of \( a^2 d - a \) is called the index of \( a \in \mathcal{A}^D \), written \( \text{ind}(a) \). We define \( \text{ind}(a) = 0 \) if \( a \in \mathcal{A}^{\text{inv}} \) and \( \text{ind}(a) = \infty \) if \( a \in \mathcal{A}^D \setminus \mathcal{A}^d \). The set of all Drazin invertible elements of \( \mathcal{A} \) will be written as \( \mathcal{A}^d \).

For the sake of brevity we will sometimes use the terms ‘d-invertible’ and ‘D-invertible’ for ‘Drazin invertible’ and ‘generalized Drazin invertible’, respectively.

The papers [11, 12] studied the invertibility of the difference and the sum of idempotents (in the setting of rings). Consequently, this topic received attention in a number of papers, extending the ordinary invertibility to the Drazin invertibility, for instance [2, 1, 4–7, 13], and the difference and sum to a linear combination of idempotents [2, 1, 4, 13]. These references by no means exhaust the literature on the topic.

For the ordinary inverse we have the following result [12, Theorem 3.5]:

\[
p - q \in \mathcal{A}^{\text{inv}} \iff 1 - pq \in \mathcal{A}^{\text{inv}} \quad \text{and} \quad p + q \in \mathcal{A}^{\text{inv}}.
\]

In this paper we highlight the fact that for Drazin and generalized Drazin invertibility we have the equivalence of the three conditions of \( p - q \in \mathcal{A}^D \), \( p + q \in \mathcal{A}^D \) and \( 1 - pq \in \mathcal{A}^D \), with a view to correcting the results of [4].

We observe that [4] states incorrectly that \( a \in \mathcal{A} \) is Drazin invertible if and only if 0 is not an accumulation point of \( \sigma(a) \). Some of the proofs use this characterization, and those results are then in fact proved for the generalized Drazin inverse rather than for the Drazin inverse. This means that some results in [4] are proved only for the generalized Drazin inverse, and the rest only for the Drazin inverse. We intend to rectify this by establishing the results in both cases.

Similarly some proofs in [5] need to be augmented by checking the finiteness of the Drazin index. This affects Theorems 3.1 and 3.2, Lemma 3.1, and Corollaries 3.1, 3.2 and 3.3 in [5], and is corrected in this paper.

We are indebted to the referee for pointing out that the main results of the paper for the generalized Drazin inverse could be obtained using the techniques of Roch–Silbermann–Gohberg–Krupnik elegantly summarised in Theorem 6.1 of the Böttcher–Spitkovsky survey paper [2]. An element \( a \in \mathcal{A} \) is generalized Drazin invertible if and only if zero is not an accumulation point of \( \sigma(a) \). Applying [2, Theorem 6.1], we can obtain a complete description of the spectrum \( \sigma(a) \) for any element \( a \) of the subalgebra \( \mathcal{B}(p, q) \) generated by \( 1, p, q \). Then \( a \in \mathcal{B}(p, q) \) is generalized Drazin invertible if and only if the preimage of zero under the determinant \( F_x(a) \) as defined in [2] is not an accumulation point of \( \sigma((p - q)^2) \). This determinant is equal to \( \alpha \beta (1 - x) \) for \( a = \alpha p + \beta q, 1 - x \) for \( a = 1 - pq \) and \( \pm x (1 - x) \) for \( a = pq = qp \). Then the main results of the paper follow.

We use alternative methods for establishing our results, while noting that the approach suggested by the referee outlined above reveals a unifying principle behind the facts.

2. Preliminary results

In this section we gather various known results we will rely on.

**Lemma 2.1** [9]. Let \( a \in \mathcal{A} \). Then \( a \in \mathcal{A}^D \) if and only if 0 is not an accumulation point of \( \sigma(a) \), and \( a \in \mathcal{A}^d \) if and only if 0 is not an essential singularity of the resolvent \( (\lambda - a)^{-1} \) of \( a \).

**Lemma 2.2.** If \( b, c \in \mathcal{A} \), then \( \sigma(bc) \setminus \{0\} = \sigma(cb) \setminus \{0\} \). Further,

\[
\begin{align*}
b c & \in \mathcal{A}^{\text{qnil}} \iff cb \in \mathcal{A}^{\text{qnil}}, \\
bc & \in \mathcal{A}^{\text{nil}} \iff cb \in \mathcal{A}^{\text{nil}}, \\
b c & \in \mathcal{A}^D \iff cb \in \mathcal{A}^D, \\
bc & \in \mathcal{A}^d \iff cb \in \mathcal{A}^d.
\end{align*}
\]

**Proof.** The spectral relation is well known; the rest follows from it on application of the preceding lemma and the equation

\[
\lambda(\lambda - cb)^{-1} = 1 + c(\lambda - bc)^{-1}b, \quad \lambda \neq 0.
\]

\( \square \)
Combining the results of [10] on isolated spectral points and Exercise VII.5.21 in [8] on poles of the resolvent (interpreted for elements of a Banach algebra in place of operators), we obtain the following result.

**Lemma 2.3.** Let \( a \in A \), let \( f \) be a function holomorphic in an open neighbourhood of \( \sigma(a) \), and let \( f^{-1}(0) \cap \sigma(a) = \{ \lambda_1, \ldots, \lambda_m \} \) (a finite set). Then

\[
\begin{align*}
    f(a) \in A^D & \iff \lambda_i - a \in A^D \quad \text{for all } i = 1, \ldots, m, \\
    f(a) \in A^d & \iff \lambda_i - a \in A^d \quad \text{for all } i = 1, \ldots, m.
\end{align*}
\]

If \( p \in A^\text{idem} \), then each element \( a \) of \( A \) has a matrix representation

\[
a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,
\]

where \( a_{ij} = p_i a p_j \), \( p_1 = p \), \( p_2 = 1 - p \).

**Lemma 2.4.** Let \( a \in A \), \( p \in A^\text{idem} \), \( \delta \in \{D, d\} \), and let

\[
a = \begin{bmatrix} b & d \\ 0 & c \end{bmatrix}_p.
\]

If two of the elements \( a, b, c \) are in \( A^\delta \), then so is the third. In each case,

\[
a^\delta = \begin{bmatrix} b^\delta & u \\ 0 & c^\delta \end{bmatrix}_p
\]

with a uniquely determined \( u \in pA(1 - p) \).

**Proof.** The case \( \delta = D \) follows from Theorem 2.3 of [3].

Let \( b \) and \( c \) be Drazin invertible. By the preceding part of the proof, \( a \in A^D \). Let \( k \in \mathbb{N} \). Then

\[
(a(1 - aa^D))^{2k} = \begin{bmatrix} (b(1 - bb^d))^k & uk \\ 0 & (c(1 - cc^d))^k \end{bmatrix}_p,
\]

with a uniquely determined \( u_k \in pA(1 - p) \). If \( k \geq \max\{\text{ind}(b), \text{ind}(c)\} \), then \((b(1 - bb^d))^k = 0 = (c(1 - cc^d))^k \) and \( (a(1 - aa^D))^{2k} = 0 \) and \( a^D = a^d \) with \( \text{ind}(a) \leq 2 \max\{\text{ind}(b), \text{ind}(c)\} \).

Next assume that \( a \) and \( c \) are Drazin invertible. Again \( b \in A^D \) by the first part of the proof. The matrix representation then ensures that \((a(1 - aa^D))^k = 0 = (c(1 - cc^d))^k \) implies \((b(1 - bb^D))^k = 0 \), that is, \( b^D = b^d \). We note that in this case \( \text{ind}(b) \leq \max\{\text{ind}(a), \text{ind}(c)\} \). The case \( a, b \in A^d \) is symmetric. \( \square \)

**Remark 2.5.** There is also a lower triangular version of the preceding lemma.

We extend Lemma 2.1 in [4] to the generalized Drazin inverse. In our proof we succeed in avoiding the use of Cline’s formula \((ba)^D = b((ab)^D)^2 a \) originally employed in [4], which would require a proof for the generalized Drazin inverse.
Lemma 2.6. Let \( a \in \mathcal{A}, p \in \mathcal{A}^{\text{idem}}, \) and let
\[
a = \begin{bmatrix}
0 & b \\
c & 0
\end{bmatrix}_p.
\]
If \( \delta \in \{D, d\} \), then \( a \in \mathcal{A}^{\delta} \) if and only if \( bc \in \mathcal{A}^{\delta} \), and in this case
\[
a^{\delta} = \begin{bmatrix}
0 & (bc)^{\delta} b \\
(cbc)^{\delta} & 0
\end{bmatrix}_p.
\]  \hspace{1cm} (2.1)

Proof. Let \( \delta = D \). Let \( bc \) be D-invertible and let \( x \) be defined by the matrix on the right hand side of (2.1). A direct calculation shows that \( ax = xa \) and \( ax^2 = x \). Write \( w = a - a^2 x \). Then
\[
w = \begin{bmatrix}
0 & u \\
v & 0
\end{bmatrix}_p, \quad w^2 = \begin{bmatrix}
uv & 0 \\
0 & vu
\end{bmatrix}_p,
\]
where \( u = b - bc(bc)^D b, v = c - cbc(bc)^D \) and \( uv = bc - (bc)^2 (bc)^D \). Since \( uv \) is quasinilpotent by the definition of \( (bc)^D \), \( vu \) is quasinilpotent by Lemma 2.2. Hence \( w^2 \), and also \( w \), is quasinilpotent, and \( x \) is the D-inverse of \( a \).

Conversely assume that \( a \) is D-invertible. Then so is
\[
a^2 = \begin{bmatrix}
bc & 0 \\
0 & cb
\end{bmatrix}_p,
\]
and \( \sigma(a^2) \setminus \{0\} = \sigma(cb) \setminus \{0\} = \sigma(cb) \setminus \{0\} \). This implies that \( bc \) and \( cb \) are D-invertible.

The case \( \delta = d \) is correctly proved in [4]. \( \Box \)

3. Linear combinations of idempotents

First we recall the result of [12] mentioned in the introduction:
\[
p - q \in \mathcal{A}^{\text{inv}} \iff 1 - pq \in \mathcal{A}^{\text{inv}} \text{ and } p + q \in \mathcal{A}^{\text{inv}}.
\]
In general, both conditions \( 1 - pq \in \mathcal{A}^{\text{inv}} \) and \( p + q \in \mathcal{A}^{\text{inv}} \) are needed to conclude that \( p - q \in \mathcal{A}^{\text{inv}} \). However, as we will show below, \( 1 - pq \in \mathcal{A}^{D} \) on its own implies \( p - q \in \mathcal{A}^{D} \). Hence we have \( p - q \in \mathcal{A}^{D} \iff p + q \in \mathcal{A}^{D} \iff 1 - pq \in \mathcal{A}^{D} \); a corresponding result also holds for the Drazin invertibility. More generally we will look at the linear combinations \( \alpha p + \beta q \) and consider their generalized Drazin and Drazin invertibility.

Lemma 3.1. Let \( p, q \in \mathcal{A}^{\text{idem}} \) and let \( \alpha, \beta \in \mathbb{C} \setminus \{0\} \). If \( \lambda \in \mathbb{C} \setminus \{0, \alpha, \beta\} \), then
\[
\lambda \in \sigma(\alpha p + \beta q) \iff (1 - \alpha^{-1} \lambda)(1 - \beta^{-1} \lambda) \in \sigma(pq).
\]  \hspace{1cm} (3.1)

Proof. Let \( \lambda \in \mathbb{C} \setminus \{0, \alpha, \beta\} \). Following [4], we observe that
\[
(1 - \alpha^{-1} \lambda - p)(\lambda - (\alpha p + \beta q))(1 - \beta^{-1} \lambda - q) = \lambda((1 - \alpha^{-1} \lambda)(1 - \beta^{-1} \lambda) - pq). \hspace{1cm} (3.2)
\]
This implies (3.1). \( \Box \)
Theorem 3.2. Let \( p, q \in \mathcal{A}^{idem} \) and let \( \alpha, \beta \in \mathbb{C} \setminus \{0\} \). If \( \delta \in \{D, d\} \), then
\[
1 - pq \in A^\delta \implies \alpha p + \beta q \in A^\delta.
\] (3.3)

Proof. Suppose that \( 1 - pq \) is D-invertible, and for a proof by contradiction assume that 0 is an accumulation point of \( \sigma(\alpha p + \beta q) \). There is a sequence \( (\lambda_n) \) in \( \sigma(\alpha p + \beta q) \) convergent to zero. According to the preceding lemma, \( \mu_n = (1 - \alpha^{-1}\lambda_n)(1 - \beta^{-1}\lambda_n) \) is a sequence in \( \sigma(pq) \setminus \{1\} \) convergent to 1; hence \( (1 - \mu_n) \) is a sequence in \( \sigma(1 - pq) \setminus \{0\} \) convergent to 0 contrary to the assumption about \( 1 - pq \). This proves \( \alpha p + \beta q \in A^D \).

Assume that \( 1 - pq \in A^d \) and write
\[
\mu = 1 - (1 - \alpha^{-1}\lambda)(1 - \beta^{-1}\lambda) = (\alpha\beta)^{-1}\lambda(\alpha + \beta - \lambda).
\]
Taking into account (3.2) and the inclusion \( \sigma(t) \subset \{0, 1\} \) valid for any idempotent \( t \), we conclude that there is \( \rho > 0 \) such that
\[
(\lambda - (\alpha p + \beta q))^{-1} = (1 - \beta^{-1}\lambda - \mu)^{-1}(1 - \alpha^{-1}\lambda - p) \quad (3.4)
\]
for all \( \lambda \) satisfying \( 0 < |\lambda| < \rho \). The resolvent \( (\mu - (1 - pq))^{-1} \) has a pole at \( \mu = 0 \). Expanding \( (1 - pq - \mu)^{-1} \) in a Laurent series at \( \mu = 0 \) and substituting \( \mu = (\alpha\beta)^{-1}\lambda(\alpha + \beta - \lambda) \), we obtain the right hand side of (3.4) as a Laurent series in \( \lambda \) with only a finite number of nonzero terms in negative powers of \( \lambda \). Thus 0 is a pole of the resolvent of \( \alpha p + \beta q \), and \( \alpha p + \beta q \) is Drazin invertible. \( \square \)

We now come to the main result which contrasts the case of the ordinary inverse. In view of the following theorem, the generalized Drazin and Drazin invertibility of the linear combination \( \alpha p + \beta q \) of idempotents are independent of the choice of scalars \( \alpha, \beta \in \mathbb{C} \setminus \{0\} \), and the case \( \alpha + \beta = 0 \) is allowed.

Theorem 3.3. Let \( p, q \in \mathcal{A}^{idem}, \alpha, \beta \in \mathbb{C} \setminus \{0\} \), and let \( \delta \in \{D, d\} \). Then
\[
\alpha p + \beta q \in A^\delta \iff 1 - pq \in A^\delta.
\]
In particular,
\[
p - q \in A^D \iff p + q \in A^D \iff 1 - pq \in A^D,
p - q \in A^d \iff p + q \in A^d \iff 1 - pq \in A^d.
\]

Proof. In view of the preceding theorem we only need to prove that \( p - q \in A^\delta \) implies \( 1 - pq \in A^\delta \). For this we could turn to the proof of Lemma 3.1 in [11]. Instead, we offer an alternative proof based on a matrix representation which will then be useful in the proof of the next theorem. Relative to the idempotent \( p \),
\[
p = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix}, \quad (p - q)^2 = \begin{bmatrix} p - q_1 & 0 \\ 0 & q_4 \end{bmatrix}, \quad 1 - pq = \begin{bmatrix} p - q_1 & -q_2 \\ 0 & 1 - p \end{bmatrix}. \quad (3.5)
\]
Clearly \( p - q \in A^\delta \) if and only if \( (p - q)^2 \in A^\delta \). The implication \( p - q \in A^\delta \implies 1 - pq \in A^\delta \) then follows from Lemma 2.4. \( \square \)

Remark 3.4. (i) Theorem 2.7 of [4] sketches a proof of the \( \delta = D \) case of the preceding theorem. However, its statement is given for \( \delta = d \), but not proved for that case. Our Theorems 3.2 and 3.3 correct this.
(ii) In [4], explicit equations for \((αp + βq)^d\) are given under various conditions on \(p\) and \(q\). In [13], Zhang and Wu also obtain explicit equations for \((αp + βq)^d\), give counterexamples to Theorems 3.2 and 3.3 in [4], and present corrected versions of these two theorems.

**Theorem 3.5.** Let \(p, q \in \mathcal{A}^{idem}\) and let

\[
\begin{align*}
\mathcal{L}_1 &= \{ p - q, \ p + q, \ 1 - p, \ p - q, \ p - q, \ 1 - pq, \ p - pq, \ p - pqp, \pq \}, \\
\mathcal{L}_2 &= \{ q - pq, \ q - p, \ 1 - q, \ q - p, \ q - pq, \ q - q, \ q - qpq \}, \\
\mathcal{L}_3 &= \{ p - q - pq, \ p - q - q, \ p - q - qpq, \ q - q - qpq \}, \\
\mathcal{K} &= \{ p + q - 1, \ 1 + p - q, \ 1 - p + q, \ pq, \ q, \ 1 - p + pqp, \ pqp \}.
\end{align*}
\]

Let \(δ \in \{D, d\}\). If one of the elements of the set \(\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3\) (resp. \(\mathcal{K}\)) is in \(\mathcal{A}^δ\), then they all are.

**Proof.** Theorem 3.3 accounts for the first three elements of \(\mathcal{L}_1\). For the rest of \(\mathcal{L}_1\) use the matrix representations as in (3.5),

\[
\begin{align*}
p - pq &= \begin{bmatrix} p - q_1 & -q_2 \\
0 & 0 \end{bmatrix}, & \quad p - q &= \begin{bmatrix} p - q_1 & 0 \\
-q_3 & 0 \end{bmatrix}, \\
1 - pqp &= \begin{bmatrix} p - q_1 & 0 \\
0 & 1 - p \end{bmatrix}, & \quad p - pqp &= \begin{bmatrix} p - q_1 & 0 \\
0 & 0 \end{bmatrix},
\end{align*}
\]

and Lemma 2.4. \(\mathcal{L}_2\) is obtained from \(\mathcal{L}_1\) by interchanging \(p\) and \(q\), and \(\mathcal{L}_3\) by replacing \(p\) by \(1 - p\) and simultaneously \(q\) by \(1 - q\) in \(\mathcal{L}_1\) (note that \((1 - p) - (1 - q) = q - p\)). \(\mathcal{K}\) is obtained from \(\mathcal{L}_1\) by replacing \(q\) by \(1 - q\). \(\square\)

Further equivalences are obtained when we interchanged \(p\) and \(q\) in \(\mathcal{K}\).

**Remark 3.6.** In Theorem 3.5 we have achieved a simplification of proofs compared to [4]. The preceding theorem accounts for, and corrects, Corollary 2.2, Theorem 2.3, Theorem 2.4, Corollary 2.3 and Corollary 2.4 of [4]. As an example, [4, Theorem 2.4] follows when in \(p - q \in \mathcal{A}^δ \iff p - pq \in \mathcal{A}^δ\) we substitute \(1 - q\) for \(q\).

**4. The commutator and anticommutator of \(p\) and \(q\)**

Given idempotents \(p\) and \(q\) in \(\mathcal{A}\), we consider the generalized Drazin and Drazin invertibility of their commutator \(pq - qp\) and anticommutator \(pq + qp\), and complete Theorems 2.5 and 2.6 of [4].

**Theorem 4.1.** Let \(p, q \in \mathcal{A}^{idem}\) and let \(δ \in \{D, d\}\). Then

\[
\begin{align*}
pq - qp &\in \mathcal{A}^δ \iff pq + qp \in \mathcal{A}^δ \iff (p - q \in \mathcal{A}^δ \text{ and } pq \in \mathcal{A}^δ).
\end{align*}
\]

**Proof.** (a) First we prove that

\[
pq - qp \in \mathcal{A}^δ \iff (p - q \in \mathcal{A}^δ \text{ and } pq \in \mathcal{A}^δ).
\]

Represent \(p\) and \(q\) by matrices as in (3.5). Then

\[
pq - qp = \begin{bmatrix} 0 & q_2 \\
-q_2 & 0 \end{bmatrix}.
\]
By Lemma 2.6, $pq - qp \in A^\delta$ if and only if $q_2q_3 \in A^\delta$. We have $q_2q_3 = pq(1 - p)qp = pqp(1 - pqp) = f(pqp)$ with $f(\lambda) = \lambda(1 - \lambda)$. Thus $pq - qp \in A^\delta \iff f(pqp) \in A^\delta$.

Since $f^{-1}(0) = \{0, 1\}$, Lemma 2.3 implies that $f(pqp) \in A^\delta$ if and only if $pqp \in A^\delta$ and $1 - pqp \in A^\delta$. The conclusion follows by Theorem 3.5 as $pqp \in A^\delta \iff pq \in A^\delta$ and $1 - pqp \in A^\delta \iff p - q \in A^\delta$.

(b) Next we show that

$$pq + qp \in A^\delta \iff (p + q \in A^\delta \text{ and } 1 - p - q \in A^\delta).$$

A straightforward check shows that $pq + qp = f(p + q)$, where $f(\lambda) = \lambda(\lambda - 1)$. Since $f^{-1}(0) = \{0, 1\}$, Lemma 2.3 says that $f(p + q) \in A^\delta$ if and only if $p + q \in A^\delta$ and $1 - (p + q) \in A^\delta$.

By Theorem 3.5, $p - q \in A^\delta \iff p + q \in A^\delta$, and $pq \in A^\delta \iff 1 - p - q \in A^\delta$. This completes the proof of the theorem. □

**Remark 4.2.** The preceding theorem completes the proofs of Theorems 2.5 and Theorem 2.6 in [4] for the case $\delta = d$, while simplifying them for the case $\delta = D$.

5. Applications to Banach space operators

Let $X$ be a Banach space and $B(X)$ the Banach algebra of all bounded linear operators on $X$. We recall that an operator $A \in B(X)$ is generalized Drazin invertible if and only if $A = A_1 \oplus A_2$, where $A_1$, $A_2$ are bounded linear operators acting on closed subspaces of $X$, with $A_1$ invertible and $A_2$ quasinilpotent. If $A_2$ is nilpotent, $A$ is Drazin invertible.

Let $P, Q \in B(X)$ be idempotent. Relative to the space decomposition $X = R(P) \oplus N(P)$, where $R(P)$ and $N(P)$ are the range and nullspace of $P$, we have

$$P = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}.$$ 

**Theorem 5.1.** If $P, Q \in B(X)$ are idempotent and $\delta \in \{D, d\}$, then the following conditions are equivalent:

(i) $I - PQ$ is $\delta$-invertible.

(ii) $\alpha P + \beta Q$ is $\delta$-invertible for any $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

(iii) $I_1 - Q_1$ is $\delta$-invertible in $B(R(P))$.

(iv) $Q_4$ is $\delta$-invertible in $B(N(P))$.

**Proof.** The equivalence of (i) and (ii) follows from Theorem 3.3. For the proof of (iii) and (iv), consider

$$I - PQ = \begin{bmatrix} I_1 - Q_1 & -Q_2 \\ 0 & I_2 \end{bmatrix}, \quad P + Q - PQ = \begin{bmatrix} I_1 & 0 \\ Q_3 & Q_4 \end{bmatrix}.$$ 

By Theorem 3.5, $I - PQ$ is $\delta$-invertible if and only if $P + Q - PQ$ is $\delta$-invertible. Lemma 2.4 modified for operator matrices then shows that the $\delta$-invertibility of $I - PQ$ is equivalent to the $\delta$-invertibility of either $I_1 - Q_1$ or $Q_4$ in the appropriate spaces. The Drazin inverse case follows similarly. □

**Remark 5.2.** If $Q \in B(X)$ is an idempotent with the operator matrix

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$$

relative to the space decomposition $X = X_1 \oplus X_2$, then the generalized Drazin or Drazin invertibility of $I_1 - Q_1$ and $Q_4$ are closely linked, in fact, one is generalized Drazin or Drazin invertible if and only
if the other is. This follows from the preceding theorem when we define \( P \) as the projection in \( \mathcal{B}(X) \) with \( R(P) = X_1 \) and \( N(P) = X_2 \).

**Remark 5.3.** (i) For projections \( P, Q \) in a Hilbert space, Böttcher and Spitkovsky [1] gave criteria for the Drazin invertibility of operators in the von Neumann algebra generated by two orthogonal projections along with explicit representations for the corresponding inverses.

(ii) Theorem 3.2 of [5] proves the equivalence of the Drazin invertibility of the difference \( P - Q \) of two Hilbert space idempotent operators with the simultaneous Drazin invertibility of \( P + Q \) and \( I - PQ \). This is now strengthened to provide the equivalence of all three conditions.

For commutator and anticommutator of \( P, Q \) we have the following result.

**Theorem 5.4.** Let \( P, Q \in \mathcal{B}(X) \) be idempotent and let \( \delta \in \{ D, d \} \). Then the following are equivalent:

(i) \( PQ - QP \) is \( \delta \)-invertible.

(ii) \( PQ + QP \) is \( \delta \)-invertible.

(iii) Both \( P - Q \) and \( PQ \) are \( \delta \)-invertible.

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**References**


