Representations of the Moore-Penrose inverse for a class of 2-by-2 block operator valued partial matrices

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We obtain necessary and sufficient conditions for 2-by-2 block operator valued triangular matrices to be Moore–Penrose (MP) invertible and give new representations of such MP inverses in terms of the individual blocks.

Keywords: positive operator; Moore–Penrose inverse; projection

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1. Introduction

The Moore–Penrose inverse (for short MP inverse) has proved helpful in systems theory, difference equations, differential equations and iterative procedures. It would be useful if these results could be extended to infinite dimensional situations. Applications could then be made to denumerable systems theory, abstract Cauchy problems, infinite systems of linear differential equations, and possibly partial differential equations and other interesting subjects (see e.g. [1,2,6,7]).

In this article, we are mainly interested in MP invertibilities and representations of the MP inverse for 2-by-2 block operator valued triangular matrices with specified properties on a Hilbert space. Let $\mathcal{H}$ and $\mathcal{K}$ be separable, infinite dimensional, complex Hilbert spaces. Denote by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For an operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $A^*$ denotes the range, the null space and the adjoint of $A$, respectively. An operator $A$ is said to be positive if $(Ax, x) \geq 0$ for all $x \in \mathcal{H}$.

For a linear manifold $\mathcal{M} \subseteq \mathcal{H}$, its closure and the orthogonal complement will be denoted by $\overline{\mathcal{M}}$ and $\mathcal{M}^\perp$, respectively. $P_{\mathcal{M}}$ will denote the projector onto $\overline{\mathcal{M}}$ along $\mathcal{M}^\perp$. The identity onto a closed subspace $\mathcal{M}$ is denoted by $I_{\mathcal{M}}$ or $I$ if there is no confusion. For $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, if there exists an operator $T^+ \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying the following four operator equations

$$TT^+ T = T, \quad T^+ T T^+ = T^+, \quad TT^+ = (TT^+)*, \quad T^+ T = (T^+ T)^*,$$

(1)

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then $T^+$ is called the MP inverse of $T$. It is well known that $T$ has the MP inverse if and only if $\mathcal{R}(T)$ is closed and the MP inverse of $T$ is unique [7,13,18]. We use $T^{(1)}$ to denote an arbitrary solution to the operator equation $TXT = T$. In general, $T^{(1)}$ is not unique and $(STK)^{(1)} = K^{-1} T^{(1)} S^{-1}$ for arbitrary invertible operators $S$ and $K$.

In recent years, representations and characterizations of the MP inverse for matrices or operators on a Hilbert space have been considered by many authors [1-4,6,7,11-21]. In this article, we are mainly interested in MP invertibilities and representations of the MP inverse for 2-by-2 block operator valued upper triangular matrices. Applying these results, we can obtain the MP inverses of 2-by-2 block operator valued matrices with specified properties.

## 2. The MP inverses of 2-by-2 block operator valued triangular matrices

In this section, we shall begin with some lemmas.

**Lemma 1** [3,4] Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ have a closed range. Then $A$ has the form

$$ A = \begin{pmatrix} A_1 & 0 \\ - & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{pmatrix}, \tag{2} $$

where $A_1$ is invertible. In this case, $A^+ = A^{-1}_1 \oplus 0$.

**Lemma 2** Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ have a closed range. If $A^{(1)}$ is one solution to the operator equation $AXA = A$, then

$$ P_{\mathcal{R}(A)} = A(A^*A)^{1/2}A^*, \quad A^+ = P_{\mathcal{R}(A^*A)}A^{(1)}P_{\mathcal{R}(A)}, $$

where $P_{\mathcal{R}(A)}$ and $A^+$ are irrespective of the choice of $(A^*A)^{1/2}$ and $A^{(1)}$, respectively.

**Lemma 3** [5,9] Let $A$ and $B$ be in $\mathcal{B}(\mathcal{H})$. Then

1. $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}((AA^* + BB^*)^{1/2})$.
2. $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A) = \mathcal{R}(AA^*)$.
3. If $A \geq 0$ is a positive operator in $\mathcal{B}(\mathcal{H})$, then $\overline{\mathcal{R}(A^{1/2})} = \overline{\mathcal{R}(A)}$.

**Lemma 4** The 2-by-2 block operator valued matrix $\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ is MP invertible if and only if $\mathcal{R}(A) + \mathcal{R}(B)$ is closed, and

$$ \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^+ = \begin{pmatrix} A^*(AA^* + BB^*)^{1/2} \\ B^*(AA^* + BB^*)^{1/2} \\ 0 \end{pmatrix}. \tag{3} $$

**Proof** Put $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$. Then $\mathcal{R}(T) = \mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(AA^* + BB^*)^{1/2}$. This implies that $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(AA^* + BB^*)$ is closed by Lemma 3. So $(AA^* + BB^*)^{1/2}$ exists if $T^+$ exists. From $T^+ = T^* (TT^*)^+$ we have

$$ T^+ = \begin{pmatrix} A^* \\ B^* \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} (AA^* + BB^*)^{1/2} \\ 0 \end{pmatrix} = \begin{pmatrix} A^*(AA^* + BB^*)^{1/2} \\ B^*(AA^* + BB^*)^{1/2} \end{pmatrix}. $$

Additionally, we include some formulae here for later use.
Corollary 5 (1) The 2-by-2 block operator valued matrix \( \begin{bmatrix} A^* & 0 \\ B & 0 \end{bmatrix} \) is MP invertible if and only if \( \mathcal{R}(A^*) + \mathcal{R}(B^*) \) is closed, and
\[
\begin{bmatrix} A^* & 0 \\ B & 0 \end{bmatrix}^+ = \begin{bmatrix} (A^*A + B^*B)^+A^*A + B^*B \sqrt{A^*A + B^*B} \\ 0 \\ 0 \end{bmatrix}.
\]

(2) The 2-by-2 block operator valued matrix \( \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \) is MP invertible if and only if \( \mathcal{R}(A) + \mathcal{R}(B) \) is closed, and
\[
\begin{bmatrix} 0 & 0 \\ B^*A \end{bmatrix}^+ = \begin{bmatrix} 0 & (AA^* + BB^*)^+ \\ B^*A \end{bmatrix}.
\]

Let \( A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{B}(\mathcal{K}) \) and \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \). We will consider the MP inverse of the 2-by-2 block operator valued upper triangular matrix \( \begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix} \) in the case that \( A \) or \( B \) is invertible.

Theorem 6 Let \( A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{B}(\mathcal{K}), C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) and \( B \) be invertible. Then, the 2-by-2 block operator valued matrix \( \begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix} \) is MP invertible if and only if \( \mathcal{R}(A) \) is closed, and
\[
\begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix}^+ = \begin{bmatrix} A^* - A^*C\triangle C^*(I - AA^*)^{-1}A^*C\triangle B^* \\ \triangle C^*(I - AA^*)^{-1} \triangle B^* \end{bmatrix},
\]
where \( \triangle = (B^*B + C^*(I - AA^*)C)^{-1} \).

Proof First, by Corollary 5, for an arbitrary invertible operator \( M, \begin{bmatrix} 0 & N \\ 0 & M \end{bmatrix} \) is MP invertible and
\[
\begin{bmatrix} 0 & N \\ 0 & M \end{bmatrix}^+ = \begin{bmatrix} 0 & 0 \\ (N^*N + M^*M)^{-1}N^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ (N^*N + M^*M)^{-1}M^* \end{bmatrix}.
\]

Second, let \( B \) be invertible. Since \( \mathcal{R}(A) \) is closed, \( \begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix} \) has the form
\[
\begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & C_1 \\ 0 & A_1 \\ 0 & B \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A) \\ \mathcal{R}(A^*) \\ \mathcal{K} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{N}(A^*) \\ \mathcal{R}(A) \\ \mathcal{K} \end{bmatrix},
\]
where \( A_1 \) as an operator from \( \mathcal{R}(A^*) \) onto \( \mathcal{R}(A) \) is invertible. Now, let \( N = (0, C_1), M = \begin{bmatrix} A_1^* & C_2 \\ 0 & B \end{bmatrix} \) and \( \triangle = (B^*B + C^*(I - AA^*)C)^{-1} = (B^*B + C_1^*C_1)^{-1} \). It is easy to check that
\[
\begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix}^+ = \begin{bmatrix} 0 & 0 \\ (N^*N + M^*M)^{-1}N^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ (N^*N + M^*M)^{-1}M^* \end{bmatrix} \]
Similar to the proof of Theorem 6, we have the following result.

**Corollary 7** Let \( A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{B}(\mathcal{H},\mathcal{K}), C \in \mathcal{B}(\mathcal{K},\mathcal{H}) \) and \( B \) be invertible. Then, the 2-by-2 block operator valued matrix \( \begin{pmatrix} A & C \\ B & 0 \end{pmatrix} \) is MP invertible if and only if \( R(C) \) is closed, and

\[
\begin{pmatrix} A^* & C \\ B^* & 0 \end{pmatrix} = \begin{pmatrix} \Delta A^*(I - CC^+) & \Delta B^* \\ C^+ - C^+A\Delta A^*(I - CC^+) & (B - C^+A)^{-1} \end{pmatrix},
\]

where \( \Delta = (B^* + A^*(I - CC^+))^{-1} \).

The generalized Schur complement \([12,21]\) plays an important role in the study of the MP invertibilities. The following theorem shows this close relation.

**Theorem 8** Assume \( \mathfrak{T} = \begin{pmatrix} A^* & C \\ C & B \end{pmatrix} \succeq 0 \) and \( R(A) \) is closed. Then \( \mathfrak{T} \) is MP invertible if and only if \( R(B - C^*A^+C) \) is closed. In this case,

\[
\begin{pmatrix} A^* & C \\ C^* & B \end{pmatrix}^{(1)} = \begin{pmatrix} A^* + A^+C(B - C^*A^+C)^+C^*A^+ & -A^+C(B - C^*A^+C)^+ \\ -(B - C^*A^+C)^+C^*A^+ & (B - C^*A^+C)^+ \end{pmatrix},
\]

\( (7) \)

\[
\begin{pmatrix} A^* & C \\ C^* & B \end{pmatrix} = \begin{pmatrix} A^* & C \\ C^* & B \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix} \begin{pmatrix} A^* & C \\ C^* & B \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix} \begin{pmatrix} A^* & C \\ C^* & B \end{pmatrix},
\]

\( (8) \)

where

\[
S_{11} = (A^2 + CC^*)^+ + (A^2 + CC^*)^+(AC + CB)S_{22}(C^*A + BC^*)(A^2 + CC^*)^+,
\]

\[
S_{12} = -(A^2 + CC^*)^+(AC + CB)S_{22},
\]

\[
S_{22} = (B^2 + C^*A - (C^*A + BC^*)(A^2 + CC^*)^+(AC + CB))^+.
\]

**Proof** Write \( \mathfrak{T} = \begin{pmatrix} A^* & C \\ C & B \end{pmatrix} \) with respect to the space decomposition \( \mathcal{H} \oplus \mathcal{K} = \mathcal{N}(A) \oplus \mathcal{R}(A) \oplus \mathcal{R}(B) \oplus \mathcal{N}(B) \), where \( A_1 \) is invertible, \( B_1 \) a is densely defined and closed injective positive operator. Since

\[
\begin{pmatrix} I_1 & -A_1^{-1}C_1 \\ -C_1A_1^{-1}I_1 & I \end{pmatrix} = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}.
\]
\( \mathbb{T} \) is MP invertible if and only if \( \mathcal{R}(B_1 - C_1 A_1^{-1} C_1) \) is closed, which is equivalent that \( \mathcal{R}(B - C^* A^* C) \) is closed. In this case,

\[
\begin{pmatrix}
A_1 & C_1 \\
C_1 & B_1
\end{pmatrix}^{(1)} = \begin{pmatrix}
I & -A_1^{-1} C_1 \\
0 & I
\end{pmatrix} \begin{pmatrix}
I & 0 \\
0 & -B_1 - C_1 A_1^{-1} C_1
\end{pmatrix}^+ \begin{pmatrix}
A_1^{-1} & 0 \\
-(-B_1 - C_1 A_1^{-1} C_1)^+ C_1 A_1^{-1} & (B_1 - C_1 A_1^{-1} C_1)^+
\end{pmatrix}.
\]

Thus, it is easy to get that

\[
\mathbb{T}^{(1)} = \begin{pmatrix}
A_1 + A_1 C_1 - C_1 A_1^{-1} C_1 & (B_1 - C_1 A_1^{-1} C_1)^+ C_1 A_1^{-1} \\
-(B_1 - C_1 A_1^{-1} C_1)^+ C_1 A_1^{-1} & (B_1 - C_1 A_1^{-1} C_1)^+
\end{pmatrix}.
\]

Let \( S_{11}, S_{12} \) and \( S_{22} \) be defined as Equation (9). If we replace \( A, B \) and \( C \) by \( A^2 + CC^*, B^2 + C^* C \) and \( AC + CB \) in the representation of \( \mathbb{T}^{(1)} \), respectively, then we have

\[
(\mathbb{T}^2)^{(1)} = \begin{pmatrix}
A^2 + CC^* & AC + CB \\
C^* A + BC^* & B^2 + C^* C
\end{pmatrix}^{(1)} = \begin{pmatrix}
S_{11} & S_{12} \\
S_{12}^* & S_{22}
\end{pmatrix}.
\]

Hence, by Lemma 2,

\[
\mathbb{T}^+ = \mathbb{T}(\mathbb{T}^2)^{(1)} \mathbb{T}(\mathbb{T}^2)^{(1)} \mathbb{T}
\]

\[
P = \begin{pmatrix}
A & C \\
C^* & B
\end{pmatrix} \begin{pmatrix}
S_{11} & S_{12} \\
S_{12}^* & S_{22}
\end{pmatrix} \begin{pmatrix}
A & C \\
C^* & B
\end{pmatrix} \begin{pmatrix}
S_{11} & S_{12} \\
S_{12}^* & S_{22}
\end{pmatrix} \begin{pmatrix}
A & C \\
C^* & B
\end{pmatrix}.
\]

The range condition in Theorem 8 is in fact a quite weak restriction on the 2-by-2 positive operator matrices. Hence, from the above theorem we can derive a variety of consequences. The following are some of them.

**Corollary 9** Suppose that the 2-by-2 positive operator matrix \( \mathbb{T} \) in Theorem 8 satisfies \( AC = CB, \mathcal{R}(A) \) and \( \mathcal{R}(B) \) are closed. Then

\[
\begin{pmatrix}
A^1 & C \\
C^* & B
\end{pmatrix}^+ = \begin{pmatrix}
A^{1/2} & 0 \\
0 & B^{1/2}
\end{pmatrix} \begin{pmatrix}
I & 0 \\
B^{1/2} C^* & I
\end{pmatrix}.
\]

**Proof** Since \( \mathbb{T} \geq 0 \), there exists a contraction operator \( T \) such that \( C = A^{1/2} T B^{1/2} \) [8] and

\[
\begin{pmatrix}
A & C \\
C^* & B
\end{pmatrix} = \begin{pmatrix}
A^{(2)} & 0 \\
0 & B^{(2)}
\end{pmatrix} \begin{pmatrix}
I & T \\
T^* & I
\end{pmatrix} \begin{pmatrix}
A^{(2)} & 0 \\
0 & B^{(2)}
\end{pmatrix}.
\]
In fact, we can choose \( T \) such that \( N(B) \subset N(T) \) and \( R(T) \subset R(A) \). From \( AC = CB \) we obtain \( A^{1/2} T = TB^{1/2} \). So \( \begin{pmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{pmatrix} \) commutes with \( \begin{pmatrix} I & T \\ I & T \end{pmatrix} \). Hence \( T^+ = \begin{pmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{pmatrix} \begin{pmatrix} I & CB \end{pmatrix} 
abla \begin{pmatrix} B = CB \end{pmatrix} \). Since \( A^{1/2} T = TB^{1/2}, C = AT = TB, T = CB^+ \) and \( T^* = B^+ C^* \). It follows

\[
T^+ = \begin{pmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{pmatrix} \begin{pmatrix} I & CB \end{pmatrix} 
\]

**Corollary 10** Suppose that the 2-by-2 positive operator matrix \( T \) in Theorem 8 satisfies \( AC + CB = 0, R(A) \) and \( R(B) \) are closed. Then

\[
\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}^+ = \begin{pmatrix} A(A^2 + CC^*)^+ & C(B^2 + C^* C)^+ \\ B^+ C^* & B \end{pmatrix}
\]

**Proof** Since \( T \geq 0 \) and \( R(A) \) is closed, \( R(A) \subset R((A, C)) \subset R(A^{1/2}) = R(A) \). It follows that \( R((I - AA^+) C(I - B^+ B)) \) is closed. Moreover, we have

\[
T^+ = T(T^2)^+ = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \begin{pmatrix} A^2 + CC^* & 0 \\ 0 & (B^2 + C^* C)^+ \end{pmatrix}
\]

**Theorem 11** Let \( A \in B(\mathcal{H}), B \in B(\mathcal{K}), C \in B(\mathcal{K}, \mathcal{H}), R(A) \) and \( R(B) \) be closed. Then the 2-by-2 block operator valued matrix \( T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \) is MP invertible if and only if \( R((I - AA^+) C(I - B^+ B)) \) is closed. Moreover, we have

\[
\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{(1)} = \begin{pmatrix} A^+ - A^+ CC_0^+ - A^+ CB + A^+ CC_0^+ CB \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix}
\]

\[
T^+ = \begin{pmatrix} A^+ & 0 \\ 0 & B^+ \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} A^* A & A^* C \\ C^* A & B^* B + C^* C \end{pmatrix} \begin{pmatrix} A^+ & 0 \\ 0 & B^+ \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & B^* \end{pmatrix}
\]

where \( C_0 = (I - AA^+)C(I - B^+ B) \).

**Proof** Since \( R(A) \) and \( R(B) \) are closed, \( T \) has the form

\[
T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} R(A^*) & N(A) \\ N(A^*) & R(B) \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & C_1 \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \begin{pmatrix} R(B^*) \\ N(B^*) \end{pmatrix} \begin{pmatrix} A_3 & 0 \\ 0 & C_3 \end{pmatrix} \begin{pmatrix} A_4 & 0 \\ 0 & C_4 \end{pmatrix} \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \begin{pmatrix} R(B^*) \\ N(B^*) \end{pmatrix}
\]
where $A_1$ and $B_1$ are invertible. Suppose that

$$S = \begin{pmatrix} I & 0 & -C_1 B_1^{-1} & 0 \\ 0 & I & 0 \\ 0 & I_1 & -C_3 B_1^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad T = \begin{pmatrix} I & 0 & -A_1^{-1} C_2 & 0 \\ 0 & I & 0 \\ 0 & I & 0 \\ 0 & I_1 & 0 & 0 \end{pmatrix}$$

Then $S$ and $T$ are invertible and $S TT = A_1 \oplus B_1 \oplus C_4 \oplus 0$. Hence $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(C_4)$ is closed, that is, $T^+$ exists if and only if $\mathcal{R}((I - AA^+)C(I - B^+ B))$ is closed. Hence, $T^{(1)}$ as an operator from $\mathcal{R}(A) \oplus \mathcal{N}(A^*) \oplus \mathcal{R}(B) \oplus \mathcal{N}(B^*)$ into $\mathcal{R}(A^*) \oplus \mathcal{N}(A) \oplus \mathcal{R}(B^*) \oplus \mathcal{N}(B)$ has the following operator matrix form

$$T^{(1)} = T(A_1^{-1} \oplus B_1^{-1} \oplus C_4^+ \oplus 0) S$$

$$= \begin{pmatrix} A_1^{-1} - A_1^{-1} C_2 C_4^+ & -A_1^{-1} C_1 B_1^{-1} + A_1^{-1} C_2 C_4^+ C_3 B_1^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_1^{-1} \\ 0 & C_4^+ & -C_4^+ C_3 B_1^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} A^+ - A^+ C C_0^+ & -A^+ C B^+ + A^+ C C_0^+ C B^+ \\ C_0^+ & B^+ - C_0^+ C B^+ \end{pmatrix},$$

where $C_0 = (I - AA^+)C(I - B^+ B)$.

Note that

$$P_{\mathcal{R}(T^*)} = T^{(1)} (TT^*)^{(1)} T = \begin{pmatrix} A^* & 0 \\ C^* & B^* \end{pmatrix} \begin{pmatrix} AA^* + C C^* CB^* \\ BC^* \end{pmatrix}^{(1)} \begin{pmatrix} A^* \\ 0 \end{pmatrix}$$

and

$$P_{\mathcal{R}(T)} = T (T^* T)^{(1)} T^* = \begin{pmatrix} A^* & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A^* A^* & A^* C \\ C^* A^* B B + C^* C \end{pmatrix}^{(1)} \begin{pmatrix} A^* \\ 0 \end{pmatrix}.$$
COROLLARY 12 Suppose that the 2-by-2 upper triangular operator matrix $T$ in Theorem 11 satisfies $AA^+C(I - B^+B - C_0^+C_0) = 0$ and $(I - AA^+ - C_0C_0^+)CB^+ B = 0$, then

$$\begin{pmatrix} A^+C_0^+ & -A^+CB^+ + A^+CC_0^+C_0^+ \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}.$$

Proof From $AA^+C(I - B^+B - C_0^+C_0) = 0$ and Equation (10) it follows that $C_2(I - C_4^+C_4) = 0$. From $(I - AA^+ - C_0C_0^+)CB^+ B = 0$ and Equation (10) it follows that $(I - C_4^+C_4)C_3 = 0$. Combining the above two equations with Equation (10, 11), we deduce that $T^{(1)}T^{(1)} = T^{(1)}$, $T^{(1)}T = I \oplus 0 \oplus I \oplus C_4^+C_4$ and $TT^{(1)} = I \oplus C_4^+C_4 \oplus I \oplus 0$, which implies that $T^{(1)}$ satisfies Equation (1). Hence

$$\begin{pmatrix} A^+C_0^+ & -A^+CB^+ + A^+CC_0^+C_0^+ \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}.$$

COROLLARY 13 Suppose that the 2-by-2 upper triangular operator matrix $T$ is given as in Theorem 11.

1. If $\mathcal{R}(A) \cap \mathcal{R}(C) = \{0\}$, then

$$\begin{pmatrix} A^+C_0^+ & -A^+CB^+ + A^+CC_0^+C_0^+ \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix},$$

where $\Delta = (B^+B + (C - C_0)^*(I - C_0C_0^+)(C - C_0))^+$.

2. If $\mathcal{R}(B^*) \cap \mathcal{R}(C^*) = \{0\}$, then

$$\begin{pmatrix} A^+C_0^+ & -A^+CB^+ + A^+CC_0^+C_0^+ \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix},$$

where $\Delta = (A^+A + (C - C_0)(I - C_0C_0^+)(C - C_0)^+)$.

3. If $\mathcal{R}(A) \cap \mathcal{R}(C) = \{0\}$ and $\mathcal{R}(B^*) \cap \mathcal{R}(C^*) = \{0\}$, then

$$\begin{pmatrix} A^+C_0^+ & -A^+CB^+ + A^+CC_0^+C_0^+ \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix},$$

Proof (1) Since $\mathcal{R}(A) \cap \mathcal{R}(C) = \{0\}$, $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, $T$ has the form

$$\begin{pmatrix} A^+C_0^+ & -A^+CB^+ + A^+CC_0^+C_0^+ \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}.$$
Then, \[
\begin{pmatrix}
A^t C \\
0 \\
0 \\
\end{pmatrix} + = \begin{pmatrix}
0 \\
0 \\
T^+; 0 \\
\end{pmatrix}, \text{ where } T = \begin{pmatrix}
0 & C_1 & C_2 \\
A_1 & 0 & 0 \\
B_1 & 0 \\
\end{pmatrix}.
\]

If we replace \(A\), \(B\) and \(C\) by \((0, C_1)\), \(A_1 \oplus B_1\) and \(C_2\) in Corollary 7, respectively, then we have

\[
T^+ = \begin{pmatrix}
0 & A_1^{-1} & 0 \\
\Delta' C_1^* (I - C_2 C_2^+) & 0 & \Delta' B_1^* \\
C_2^+ C_1 \Delta' C_1^* (I - C_2 C_2^+) & 0 & -C_2^+ C_1 \Delta' B_1^* \\
\end{pmatrix},
\]

where \(\Delta' = (B_1^* B_1 + C_1^* (I - C_2 C_2^+)) C_1\). Hence

\[
\begin{pmatrix}
A^t C \\
0 & B \\
\end{pmatrix} + = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & A_1^{-1} & 0 & 0 \\
\Delta' C_1^* (I - C_2 C_2^+) & 0 & \Delta' B_1^* & 0 \\
C_2^+ C_1 \Delta' C_1^* (I - C_2 C_2^+) & 0 & -C_2^+ C_1 \Delta' B_1^* & 0 \\
\end{pmatrix} + \begin{pmatrix}
0 \\
\Delta' C_1^* (I - C_2 C_2^+) \\
-\Delta' B_1^* \\
\Delta' C_1^* (I - C_2 C_2^+) \\
\end{pmatrix},
\]

where \(\Delta = (B^* B + (C - C_0)^* (I - C_0 C_0^+)) (C - C_0)^+\), \(C_0 = (I - AA^+) (I - B^+ B)\).

Similar to the proof of (1), we can prove (2) and (3), so the details are omitted. ■

Finally, we consider the MP inverse of the 2-by-2 block operator valued upper triangular matrix \(T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\) in the case that \(\mathcal{R}(A)\) is closed.

**Theorem 14** Let \(A \in B(\mathcal{H}), B \in B(\mathcal{K}), C \in B(\mathcal{K}, \mathcal{H})\) and \(\mathcal{R}(A)\) be closed. Then the 2-by-2 block operator valued matrix \(T\) is MP invertible if and only if \(\mathcal{R}(C^* (I - AA^+) + \mathcal{R}(B^+)\) is closed, and

\[
\begin{pmatrix}
A^t C \\
0 & B \\
\end{pmatrix}^{(1)} = \begin{pmatrix}
A^+ - A^+ C (B^* B + C^* (I - AA^+))^{-1} A^* C^* (I - AA^+) C^+ \end{pmatrix}^{(B^* B + C^* (I - AA^+) C^+) B^+}
\]

Moreover, if \(\mathcal{R}(A) \cap \mathcal{R}(C) = \{0\}\), then

\[
\begin{pmatrix}
A^t C \\
0 & B \\
\end{pmatrix}^{+} = \begin{pmatrix}
A^+ \\
(B^* B + C^* (I - AA^+) C^+) B^+ \\
\end{pmatrix},
\]
Proof Since $\mathcal{R}(A)$ is closed, we have

\[
\begin{pmatrix}
A_1^1 & C_1 \\
0 & B
\end{pmatrix} = \begin{pmatrix}
A_1 & 0 \\
0 & C_2 \\
0 & 0 & B
\end{pmatrix} : \begin{pmatrix}
\mathcal{R}(A^*) \\
\mathcal{N}(A) \\
\mathcal{K}
\end{pmatrix} \rightarrow \begin{pmatrix}
\mathcal{R}(A) \\
\mathcal{N}(A^*) \\
\mathcal{K}
\end{pmatrix}.
\] (13)

Then $A_1$ is invertible and

\[
\begin{pmatrix}
A_1 & 0 & C_1 \\
0 & C_2 \\
0 & 0 & B
\end{pmatrix} = \begin{pmatrix}
A_1 & 0 & 0 \\
0 & 0 & C_2 \\
0 & 0 & 0 & B
\end{pmatrix}.
\]

It shows that $T$ has the MP inverse if and only if $\mathcal{R}((C_2^*, B^*))$ is closed, i.e. $\mathcal{R}(C^*(I - AA^*)) + \mathcal{R}(B^*)$ is closed. So

\[
\begin{pmatrix}
A_1 & C_1 \\
0 & B
\end{pmatrix}^{[1]} = \begin{pmatrix}
I & 0 & -A_1^{-1}C_1 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix} \begin{pmatrix}
A_1 & 0 & C_1 \\
0 & C_2 \\
0 & 0 & B
\end{pmatrix} = \begin{pmatrix}
A_1^{-1} & 0 & 0 \\
0 & -A_1^{-1}C_1 & 0 \\
0 & C_2 & B^*
\end{pmatrix}
\]

If $\mathcal{R}(A) \cap \mathcal{R}(C) = \{0\}$, then $C_1 = 0$ in Equation (13). So

\[
\begin{pmatrix}
A_1 & C_1 \\
0 & B
\end{pmatrix}^+ = \begin{pmatrix}
A_1^{-1} & 0 \\
0 & K^+
\end{pmatrix} \text{ (where } K = \begin{pmatrix}
0 & C_2 \\
0 & B
\end{pmatrix})
\]

\[
= \begin{pmatrix}
A_1^{-1} & 0 & 0 \\
0 & -A_1^{-1}C_1 & 0 \\
0 & C_2 & B^*
\end{pmatrix}
\]

\[
A_1^+ - A^+C(B^*B + C^*(I - A^+))A_1^{[1]} - A_1^{-1}C_1(C_2^*C_2 + B^*B)^+B^*
\]

\[
= \begin{pmatrix}
A_1^+ & -A^+C(B^*B + C^*(I - A^+)C)^+B^* \\
A_1^{-1} & -A_1^{-1}C_1(C_2^*C_2 + B^*B)^+B^*
\end{pmatrix}
\]

\[
(\mathcal{R}(B^*) + \mathcal{R}(B^* + C^*(I - A^+)C))B^*_*
\]
In Campbell and Meyer’s book 3, they have stated that the MP inverse of an upper block triangular matrix $T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is still an upper block triangular if and only if $\mathcal{R}(C) \subset \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subset \mathcal{R}(B^*)$ (See Theorem 3.4.1). We can show this result holds in the infinite dimensional case and is a special case of Theorem 14.

**COROLLARY 15** [3, Theorem 3.4.1] Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$, $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $\mathcal{R}(A)$ and $\mathcal{R}(B)$ be closed. Then $T = T^* = \begin{pmatrix} A^+ & -A^*C^+B^+ \\ 0 & B^+ \end{pmatrix}$ if and only if $\mathcal{R}(C) \subset \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subset \mathcal{R}(B^*)$.

**Proof** ‘Sufficiency’ since $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, $T$ is MP invertible by Theorem 14. If $\mathcal{R}(C) \subset \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subset \mathcal{R}(B^*)$, then $(I - AA^+)C = 0$ and $C(I - B^+B) = 0$. By Theorem 14, we have

\[
T^{(1)} = \begin{pmatrix} A^+ - A^+C(B^+B + C^*(I - A^+)C)^+B^* \\ AA^+C^*(I - A^+) + (B^+B + C^*(I - A^+)C)^+B^* \\ 0 & B^+ \end{pmatrix}.
\]

Note that $TT^{(1)} = AA^+ \oplus BB^+$, $T^{(1)}T = A^+A \oplus B^+B$ and $T^{(1)}TT^{(1)} = T^{(1)}$. So $T^{(1)} = T^+$.

‘Necessity’ since $TT^+ = \begin{pmatrix} AA^+ & -AA^+CB^+ + CB^+ \\ 0 & BB^+ \end{pmatrix}$ and $T^+T = \begin{pmatrix} A^+A & -A^+CB^+B + A^+C \\ 0 & B^+B \end{pmatrix}$ are selfadjoint, we have $-AA^+CB^+ + CB^+ = 0$ and $-A^+CB^+B + A^+C = 0$. From $TT^+T = T$, we have $C = AA^+C = CB^+B$. Hence $\mathcal{R}(C) \subset \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subset \mathcal{R}(B^*)$.

**Remark** (1) If we assume in addition that $\mathcal{R}(C) \subset \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subset \mathcal{R}(B^*)$, then $T$ satisfies Theorem 11, Corollary 12 and $C_0 = (I - AA^+)C(I - B^+B) = 0$. Thus Theorem 3.4.1 in [3] is also a special case of Theorem 11 and Corollary 12.

(2) In this article, we investigate the MP inverse of 2-by-2 block operator valued triangular matrices and give representations of such MP inverses in terms of the individual blocks. The results are new even for the finite dimensional case. It is natural to ask if we can extend our results to the 2-by-2 block operator valued matrices and weighted MP inverse in the Hilbert space [10–14], which will be the future research topic.

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