The generalized inverse of upper triangular operator matrix and its applications in perturbation analysis

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Abstract. In this paper, we investigate the representations of generalized inverse and generalized Drazin inverse of upper triangular block operator matrix on Banach spaces. We consider some applications of our results to the perturbation and analyze a number of special cases.

Keywords: generalized inverse, Drazin inverse, Perturbation

1. Introduction

The generalized inverse (for short G-inverse) and the generalized Drazin inverse (for short GD-inverse) have proved helpful in systems theory, difference equations, differential equations and iterative procedures. Let \( X \) and \( K \) be separable, infinite dimensional, complex Banach spaces. Denote by \( B(X, K) \) the set of all bounded linear operators from \( X \) into \( K \). For an operator \( A \in B(X, K) \), \( R(A), N(A) \) denote the range, the null space and the adjoint of \( A \), respectively. For \( T \in B(X, K) \), if there exists \( T^+ \in B(K, H) \) satisfying the following four operator equations

\[
TT^+T = T^+T T, \quad T^+TT^+ = T^+, \quad T^+ = (TT^+)\text{,} \quad T = (T^+T)^+,
\]

then \( T^+ \) is called the G-inverse of \( T \). It is well known that \( T \) has the G-inverse if and only if \( RT \) is closed and the G-inverse of \( T \) is unique (see [1, 4, 6]). The concept of the GD-inverse in a Banach algebra was introduced by Kolliha[5], which is the element \( T^d \in B(X) \) such that

\[
T^dT^d = T^d, \quad TT^d = T^dT \quad \text{and} \quad T - T^dT \text{ is quasinilpotent.}
\]

Let \( P = I - TT^d \). The matrix form of \( T \) with respect to the space decomposition \( X = N(P) \oplus R(P) \) is given by \( T = T_1 \oplus T_2 \), where \( T_1 \) is invertible and \( T_2 \) is quasinilpotent.

In this paper, we are mainly interested in G (or GD)-invertibilities and representations of the G (or GD)-inverse for 2 by 2 block operator valued triangular matrices. Using the technique of block operator matrices, we will investigate the explicitly representations of the G (or GD)-inverse under suitable conditions. Our results are new and some recent results are extended.

2. Main results and its proofs

In this part, we get the following useful results.

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**Theorem 1.** Let \( A \in B(X), B \in B(K), C \in B(K, X) \) and \( B \) be invertible. Then the 2 by 2 block operator valued matrix 

\[
A \begin{pmatrix}
C \\
0
\end{pmatrix} B = \begin{pmatrix}
A' - A'C\Delta C'(I - AA') & -A'C\Delta B' \\
\Delta C'(I - AA') & \Delta B'
\end{pmatrix}, 
\]

where \( \Delta = (B' + C'(I - AA'))C^{-1} \).

**Proof.** Since

\[
\begin{pmatrix}
A' \\
C'
\end{pmatrix} = \begin{pmatrix}
I \\
C' B'
\end{pmatrix}^{-1} \begin{pmatrix}
I \\
0
\end{pmatrix}.
\]

\( R(T^-) \) is closed if and only if \( R(A') \). This shows that \( T \) is G-invertible if and only if \( R(A) \) is closed. In this case \( T \) has the form

\[
\begin{pmatrix}
A \\
0
\end{pmatrix} = \begin{pmatrix}
0 & C_1 \\
0 & A_2
\end{pmatrix} \begin{pmatrix}
N(A) \\
R(A') \\
K
\end{pmatrix},
\]

where \( A_i \) as an operator from \( R(A') \) onto \( R(A) \) is invertible. Now let \( N = \begin{pmatrix} 0 \\ C_1 \end{pmatrix}, \ M = \begin{pmatrix} A \\ C_2 \\ B \end{pmatrix} \) and \( \Delta = (B' + C'(I - AA'))B^{-1} = (B' + C_1C_1)^{-1} \). It is easy to check that

\[
\begin{pmatrix}
A \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
N \\
M
\end{pmatrix} \begin{pmatrix}
0 \\
N \\
M
\end{pmatrix} = \begin{pmatrix}
0 \\
N^*N + M^*M
\end{pmatrix} \begin{pmatrix}
0 \\
N^*N + M^*M
\end{pmatrix} \begin{pmatrix}
0 \\
N^*N + M^*M
\end{pmatrix}.
\]

Remark. (1) In Theorem 1, if \( R(B) \) is closed, we can show that \( T \) is G-invertible if and only if \( R((I - AA')(I - B'B)) \) is closed in a similar way. In this case, \( T^- \) has a very complex representation. But we can show that inner inverse \( T^I \) has the form as

\[
\begin{pmatrix}
A \\
0
\end{pmatrix} = \begin{pmatrix}
A^* - A'C\Delta C'(I - AA') & -A'C\Delta B' \\
\Delta C'(I - AA') & \Delta B'
\end{pmatrix},
\]

where \( C_0 = (I - AA')C(I - B'B) \). In addition, if \( AA'C(I - B'B - C_0C_0) = 0 \) and \( (I - B'B - C_0C_0)C_0B = 0 \), a direct calculation can show that

\[
\begin{pmatrix}
A \\
0
\end{pmatrix} = \begin{pmatrix}
A^* - A'C\Delta C' & -A'C\Delta B' \\
\Delta C' & \Delta B'
\end{pmatrix}.
\]

(2) If we assume as well that \( R(C) \subset R(A) \) and \( R(C) \subset R(B^-) \), then \( T \) satisfies remark (1), and \( C_0 = (I - AA')C(I - B'B) = 0 \). Then we have

\[
\begin{pmatrix}
A \\
0
\end{pmatrix} = \begin{pmatrix}
A^* & -A'C\Delta B' \\
0 & B^-
\end{pmatrix}.
\]

Thus Theorem 3.4.1 in [1] is a special case of our results.

**Theorem 2.** Let \( A_i, B_i \in B(X), \ A_2, B_2 \in B(K), \ B_3 \in B(K, X) \) such that \( A_i \) invertible and \( A_2 \) is
quasinipotent, \( \|A^{-1}B\| < 1 \) and \( A_2B_2 = B_2A_2 \). If \( B_2 \) is GD-invertible, then the sum of 2 by 2 operator valued matrix \( A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \) and \( B = \begin{pmatrix} B_1 & B_3 \\ 0 & B_2 \end{pmatrix} \) is GD-invertible. \((A + B)^t = \begin{pmatrix} (A_1 + B_1)^{-1} & S \\ 0 & (A_2 + B_2)^t \end{pmatrix} \),

Where
\[
S = \sum_{n=0}^{\infty} (A_1 + B_1)^{-2} B_1 (A_2 + B_1)^t [I - (A_1 + B_1) B_1 (A_2 + B_1)^t]^n (A_1 + B_1)^{-1} B_1 B_2^t \sum_{n=0}^{\infty} (B_2^t)^{-1} (A_2)^t .
\]

Proof. Since \( A \) is GD-invertible, \( A \) and \( B \) have the form \( A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \) and \( B = \begin{pmatrix} B_1 & B_3 \\ 0 & B_2 \end{pmatrix} \), With \( A_1 \) is invertible and \( A_2 \) is quasinipotent. \((A + B)^t \) has the form \((A_1 + B_1)^{-1} S \begin{pmatrix} (A_2 + B_2)^t \end{pmatrix} \), Where
\[
S = \sum_{n=0}^{\infty} (A_1 + B_1)^{-2} B_1 (A_2 + B_1)^t [I - (A_1 + B_1) B_1 (A_2 + B_1)^t]^n (A_1 + B_1)^{-1} B_1 B_2^t \sum_{n=0}^{\infty} (B_2^t)^{-1} (A_2)^t .
\]

Hence the proof is completed.

3. Some Applications in perturbation analysis

Let \( T \in B(X, K) \) and let \( \bar{T} = T + \delta T \) be the small perturbation of \( T \) by \( \delta T \in B(X, K) \). The perturbation theory of \( G \) (or GD)-inverse is concerned with the question that if \( T \) is \( G \) (or GD)-invertible, then when is \( T \) is \( G \) (or GD)-invertible? What are the upper bounds of \( \|T + \delta T\| \) and \( \|\delta T\| \)? From Theorem 1, a straightforward computation can obtain the following perturbation result.

**Theorem 3.** Suppose that \( \|T^+ \delta T\| < 1 \), and \( \delta TT^+ = TT^+ \delta T \). Then \( \bar{T}^+ \) is well defined if and only if \( R((I - TT^+) \delta T) \) is closed. In this case,
\[
\bar{T}^+ = \left[ (I + (I + T^+ \delta T)^{-1} T^+ \delta T(I - T^+ T)(I - (I - TT^+) \delta T)) \right] T^+ T \times \left[ \left[ (I + T^+ \delta T)^{-1} T^+ \delta T(I - T^+ T)(I - (I - TT^+) \delta T)) \right] \right]^{-1} (I + T^+ \delta T)^{-1} T^+ \delta T(I - T^+ T)
\]

In Theorem 3, if we suppose that \( R(\delta T) \subset R(T) \), this is equivalent to that \( B = 0 \) in Theorem 1, then \( (I - TT^+) \delta T \) is injective with closed range, then \( TT^+ = I \) in Corollary 1. So the following corollary can be obtained immediately.

**Corollary 4.** [6] Suppose that \( \|T^+ \delta T\| < 1 \), and \( \delta TT^+ = TT^+ \delta T \). Then \( \bar{T}^+ \) is well defined,
\[
\bar{T}^+ = \left[ (I + (I + T^+ \delta T)^{-1} T^+ \delta T(I - T^+ T)(I - (I - TT^+) \delta T)) \right] T^+ T \times \left[ \left[ (I + T^+ \delta T)^{-1} T^+ \delta T(I - T^+ T)(I - (I - TT^+) \delta T)) \right] \right]^{-1} (I + T^+ \delta T)^{-1} T^+ \delta T(I - T^+ T)
\]

In particular, if we suppose that \( T \in B(X, K) \) is injective with closed range, then \( TT^+ = I \). The following corollary is the special case of Theorem 3 and Corollary 4.

**Corollary 5.** [4] Suppose that \( T \in B(X) \) is injective with closed range. If \( R(\delta T) \subset R(T) \) and \( \|T^+ \delta T\| < 1 \), then \( \bar{T}^+ \) is injective with closed range. Moreover,
\[
\bar{T}^+ = (I + T^+ \delta T)^{-1} T^+ = T^+ (I + \delta TT^+)^{-1} \text{ and } \|T^+ \delta T\| \leq \|T^+ \delta T\| T^+ .
\]
From Theorem 2, we can also get some results for the perturbation of generalized Drazin inverse. 

**Theorem 6.** Suppose that \( \| \delta T T^d \| < 1 \), \( T^* \delta T(I - T^*) = 0 \) and \( T^* T \delta T = T^* \delta T T \). If \( T^* \delta T \) is GD-invertible, then \( \tilde{T} \) is GD-invertible. In this case, 
\[
\tilde{T}^d = (1 + T^d \delta T)^{-1} T^d + (1 + T^d \delta T)^{-1} (I - TT^d) \sum_{n=0}^{\infty} (T^d \delta T)^n (-T)^n \\
+ \left[ \sum_{n=0}^{\infty} \left( (I + T^d \delta T)^{-1} T^d \right)^{n+2} \right] T^d \delta T (I - TT^d) (T + \delta T)^n \\
\times (I - (T + \delta T)(I - TT^d) \sum_{n=0}^{\infty} (T^d \delta T)^n (-T)^n ).
\]

In Theorem 2, if we set \( B_1 = 0 \) and \( B \) is quasinilpotent, which equivalent to \( \delta TT^d = 0 \) and \( \delta T \) is quasinilpotent, then we have the following perturbation result.

**Corollary 7.**[3] Let \( T \in B(X) \) be GD-invertible such that \( T^* \delta T(I - T^*) = 0 \) and \( T^* T \delta T = T^* \delta T T \). If \( \delta TT^d = 0 \) and \( \delta T \) is quasinilpotent, then
\[
\tilde{T}^d = \sum_{n=0}^{\infty} (T^d \delta T)^n (T + \delta T)^n + T^d.
\]

In particular, if we set \( A_B B_1 = A_B \), \( B_1 = 0 \) and \( T^* \delta T \) is quasinilpotent, which equivalent to the \( TT^d T = T^* \delta T T \), \( T^* \delta T = \delta TT^d \) and \( T^* \delta T \) is quasinilpotent, then we have the following perturbation result.

**Corollary 8.**[2] Suppose that \( \| \delta TT^d \| < 1 \), \( T^* \delta T(I - T^*) = 0 \) and \( T^* T \delta T = T^* \delta T T \). If \( T^* \delta T = \delta TT^d \), \( \sigma(T^* \delta T) = \{0\} \), then we have that
\[
\tilde{T}^d = (1 + T^d \delta T)^{-1} T^d = T^d (I + \delta TT^d)^{-1}.
\]

\[
\frac{\| \tilde{T}^d - T^d \|}{\| T^d \|} \leq \frac{\| T^d \delta T \|}{1 - \| T^d \delta T \|} \text{ and } \frac{\| T^d \delta T \|}{1 + \| T^d \delta T \|} \leq \frac{\| \tilde{T}^d \|}{\| T^d \delta T \|}.
\]

4. **Concluding remarks**

In this paper, we investigate the perturbation of the G- invertible and GD-invertible operators and derive explicit G-inverse and GD-inverse expressions for the perturbations under certain restrictions on the perturbing operators. It is natural to ask if we can extend our results to the W-weighted G-inverse and W-weighted GD-inverse, which will be the future research topic.

References


