A NEW CHARACTERIZATION OF THE CLOSEDNESS OF THE SUM OF TWO SUBSPACES

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Abstract  In this article, a new characterization of the closedness of the sum of two closed subspaces of a Hilbert space is established.

Key words  Subspace, projection, sum of two closed subspaces

2000 MR Subject Classification  47A10, 47A55

1 Introduction

The closedness of the sum of two subspaces, which is a closed relation with the projections on these subspaces, plays a relevant role in a variety of least-square problems. As a sample of their applications and of their relatives, namely, generalized inverses, they have been used in optimization (feasibility theory, interior point methods), statistics (linear regression, weighted estimation), and signal processing (noise reduction).

In a series of articles, D. T. Kato[2], R. Hagen, S. Roch and B. Silbermann[3], R. Piziak, P. L. Odell and R. Hahn[4], I. Spitkovsky[7] studied the closedness of the subspace and the projection operators. The reader may refer to the book by D. T. Kato[2] for excellent surveys on the history and motivations of the problem. In this article we present a different approach to the closedness of the sum of two subspaces. Based on technique and results on projections and generalized inverse, a new characterization of the closedness of the sum of two closed subspaces of a Hilbert space is established. These results are important in the study on weighted projection (usually, operator norms, vector norms and angles) and generalized inverses (Moore-Penrose inverse and Drazin inverse).

Let $\mathcal{B}(\mathcal{H})$ denote the set of all linear bounded operators on a Hilbert space $\mathcal{H}$. For an operator $A \in \mathcal{B}(\mathcal{H})$, the symbols $A^*$, $A^+$, and $\mathcal{R}(A)$ will stand for the adjoint, the Moore-Penrose inverse, and the range of $A \in \mathcal{B}(\mathcal{H})$ respectively. It is well known that (see [3])

$$
(A^+)^* = (A^*)^+, (AA^*)^+ = (A^*)^+A^+, A^+ = A^*(AA^*)^+, \text{ and } A^+ = (A^+A)^+A^*. \quad (1)
$$

*Received August 26, 2005; revised July 10, 2006*
For two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$, the quantity $\gamma(\mathcal{M}, \mathcal{N})$ between $\mathcal{M}$ and $\mathcal{N}$ (see [2]) is defined by

$$\gamma(\mathcal{M}, \mathcal{N}) = \inf_{x \in \mathcal{M}, \, x \notin \mathcal{N}} \frac{\text{dist}(x, \mathcal{N})}{\text{dist}(x, \mathcal{M} \cap \mathcal{N})}. \quad (2)$$

By this formula $\gamma(\mathcal{M}, \mathcal{N})$ is defined only when $\mathcal{M}$ is not a subspace of $\mathcal{N}$. If $\mathcal{M} \subseteq \mathcal{N}$ we set $\gamma(\mathcal{M}, \mathcal{N}) = 1$. Obviously $\gamma(\mathcal{M}, \mathcal{N}) = 1$ if $\mathcal{M} \supseteq \mathcal{N}$. Denote

$$\bar{\gamma}(\mathcal{M}, \mathcal{N}) = \min\{\gamma(\mathcal{M}, \mathcal{N}), \gamma(\mathcal{N}, \mathcal{M})\} \quad (3)$$

and call it the minimum gap between $\mathcal{M}$ and $\mathcal{N}$.

For two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$, denote $\mathcal{H}_1 = \mathcal{M} \cap \mathcal{N}$, $\mathcal{H}_2 = \mathcal{M} \cap \mathcal{N}^\perp$, $\mathcal{H}_3 = \mathcal{M}^\perp \cap \mathcal{N}$, $\mathcal{H}_4 = \mathcal{M}^\perp \cap \mathcal{N}^\perp$ and let $\mathcal{H}_5 = \mathcal{M} \cap (\mathcal{H} \ominus (\bigoplus_{i=1}^{4} \mathcal{H}_i))$ and $\mathcal{H}_6 = (\mathcal{H} \ominus (\bigoplus_{i=1}^{4} \mathcal{H}_i)) \ominus \mathcal{H}_5$. It is clear that $\mathcal{H}_i \perp \mathcal{H}_j$, $i \neq j$ and $1 \leq i, j \leq 6$. If $P_M$ and $P_N$ are orthogonal projections onto $\mathcal{M}$ and $\mathcal{N}$ respectively, we have the following lemma, which is useful later.

**Lemma 1** (see [1]). Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$, then $P_M \mathcal{H}_i \subseteq \mathcal{H}_i$ and $P_N \mathcal{H}_i \subseteq \mathcal{H}_i$, $1 \leq i \leq 4$, and $P_M$ and $P_N$ have the following operator matrices

$$P_M = I \oplus I \oplus 0 \oplus 0 \oplus \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad (4)$$

$$P_N = I \oplus I \oplus I \oplus 0 \oplus \begin{pmatrix} Q & Q^\perp(I - Q)^\perp D \\ D^*Q^\perp(I - Q)^\perp & D^*(I - Q)D \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = \bigoplus_{i=1}^{6} \mathcal{H}_i$, respectively, where $Q$ is a positive contraction on $\mathcal{H}_5$, and 0 and 1 are not eigenvalues of $Q$ and $D$ is a unitary operator from $\mathcal{H}_6$ onto $\mathcal{H}_5$.

**Proof** Without loss of generality, we can set $\mathcal{H}_i = 0$ for $i = 1, 2, 3, 4$. Denote by $P_M$ and $P_N$ the projections onto $\mathcal{M}$ and $\mathcal{N}$ respectively. Since a $2 \times 2$ operator matrix $T = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \geq 0$ if and only if $A \geq 0, B \geq 0, C = D^*$ and there exists a contraction $S$ such that $C = A^{\frac{1}{2}}SB^{\frac{1}{2}}$, $P_M$ and $P_N$ have the following operator matrix forms

$$P_M = \begin{pmatrix} I_M & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_N = \begin{pmatrix} Q_{11} & Q_{11}^\perp DQ_{22}^\perp \\ Q_{22}^\perp D^*Q_{11}^\perp & Q_{22} \end{pmatrix} \quad (5)$$

with respect to the space decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where $Q_{11}$ and $Q_{22}$ are positive contractions on $\mathcal{M}$ and $\mathcal{M}^\perp$ respectively. $D$ is a contraction from $\mathcal{M}^\perp$ into $\mathcal{M}$. Since $P_N = P_N$,

$$\begin{pmatrix} Q_{11}^\perp + Q_{11}^\perp DQ_{22}D^*Q_{11}^\perp & Q_{11}^\perp DQ_{22}^\perp + Q_{22}^\perp DQ_{22}^\perp \\ Q_{22}^\perp D^*Q_{11}^\perp + Q_{22}^\perp D^*Q_{11}^\perp & Q_{22}^\perp D^*Q_{11}DQ_{22}^\perp + Q_{22}^\perp \\
\end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{11}^\perp DQ_{22}^\perp \\ Q_{22}^\perp D^*Q_{11} & Q_{22} \\
\end{pmatrix}.$$
Observing that the regularity of \((M, N)\) implies that \(Q_{11}, Q_{22}, I_M - Q_{11}\) and \(I_M - Q_{22}\) are injective and solving the above system of equations, we have

\[
DD^* = I_M, \quad D^* D = I_{M^\perp}, \quad \text{and} \quad Q_{22} = D^*(I_M - Q_{11}) D.
\]

Take \(Q = Q_{11}\), then \(Q\) and \(I_M - Q\) are injective, so neither 0 nor 1 is in \(\sigma_p(Q)\). Thus for every closed subspaces \(M\) and \(N\), it implies

\[
P_M = I \oplus I \oplus 0 \oplus 0 \oplus I \oplus Q\]

\[
P_N = I \oplus 0 \oplus I \oplus 0 \oplus \left( Q D^* Q^\perp (I - Q)^{\frac{1}{2}} \quad Q^\perp (I - Q)^{\frac{1}{2}} D \right)
\]

with respect to the space decomposition \(H = \bigoplus_{i=1}^6 H_i\).

### 2 Geometric Structure of \(\gamma(M, N)\)

If \(P_M\) and \(P_N\) have the operator matrices (4), for \(x \in M\), we have \(x = x_1 + x_2 + x_5\) with \(x_i \in H_i\), \(i = 1, 2, 5\), and \(P_N x = x_1 + Q x_5 + D^* Q^{\frac{1}{2}} (I - Q)^{\frac{1}{2}} x_5\). Observing that dist\((x, N)\) is \(\|x\|^2 - \|P_N x\|^2\), then

\[
\text{dist}(x, N)^2 = \|x_1\|^2 + \|x_2\|^2 + \|x_5\|^2 - \|x_1\|^2 - \|Q x_5\|^2 - \|D^* Q^{\frac{1}{2}} (I - Q)^{\frac{1}{2}} x_5\|^2
\]

and

\[
\text{dist}(x, M \cap N)^2 = \|x_1\|^2 + \|x_2\|^2 + \|x_5\|^2 - \|x_1\|^2 = \|x_2\|^2 + \|x_5\|^2.
\]

**Proposition 2** Let \(M\) and \(N\) be two closed subspaces of \(H\), and \(P_M\) have the operator matrices (4). Then \(\gamma(M, N) \leq \|(I - Q)^{\frac{1}{2}}\||\).

**Proof** By Lemma 1, \(0 \leq Q \leq I\), then \(0 \leq I - Q \leq I\). So \(I - Q\) is also a positive contraction. For a vector \(x \in M\), \(x\) has the decomposition \(x = x_1 + x_2 + x_5\) with \(x_i \in H_i, i = 1, 2, 5\), then

\[
\gamma(M, N) = \inf_{x \in M, x \in N} \frac{\text{dist}(x, N)}{\text{dist}(x, M \cap N)}
\]

\[
= \inf_{x \in M, x \in N} \sqrt{\frac{\|x_2\|^2 + \|(I - Q)^{\frac{1}{2}} x_5\|^2}{\|x_2\|^2 + \|x_5\|^2}}
\]

\[
= \inf_{x \in M, x \in N} \frac{\|(I - Q)^{\frac{1}{2}} x_5\|}{\|x_5\|}
\]

\[
\leq \|(I - Q)^{\frac{1}{2}}\|.
\]

Let \(\hat{M} = M \oplus (M \cap N)\) and \(\hat{N} = N \oplus (M \cap N)\), then \(\hat{M} \cap \hat{N} = 0\). By Proposition 2, the following consequences are clear.

**Corollary 3** If \(M\) and \(N\) are two closed subspaces of \(H\), then \(\gamma(M, N) = \gamma(\hat{M}, \hat{N})\).

The next lemma is quite useful.

**Lemma 4** [5] Let \(A\) and \(B\) be in \(B(H)\). Then the following statements hold:

(1) \(R(A) = R((AA^*)^{\frac{1}{2}});\)
(2) $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}((AA^* + BB^*)^{1/2})$;

(3) The range $\mathcal{R}(A)$ of $A$ is closed if and only if $\mathcal{R}(A) = \overline{\mathcal{R}(A)}$;

(4) If $A$ is a positive operator in $\mathcal{B}(\mathcal{H})$, then $\overline{\mathcal{R}(A^+)} = \overline{\mathcal{R}(A)}$, where $\overline{K}$ denotes the closure of the set $K$.

**Proposition 5** If $\mathcal{M}$ and $\mathcal{N}$ are two closed subspaces of $\mathcal{H}$, then $\tilde{\gamma}(\mathcal{M}, \mathcal{N}) = \gamma(\mathcal{M}, \mathcal{N}) = \gamma(\mathcal{N}, \mathcal{M})$.

**proof** Let $P_\mathcal{M}$ and $P_\mathcal{N}$ have the operator matrices as the formula (4), since $\mathcal{M} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_5$, $\mathcal{N} = \mathcal{H}_1 \oplus \mathcal{H}_3 \oplus R(P_\mathcal{N}|_{\mathcal{H}_3 \oplus \mathcal{H}_6})$, and

$$
\begin{pmatrix}
Q \\
D^*Q^{1/2}(I-Q)^{1/2}
\end{pmatrix}
\begin{pmatrix}
Q & Q^{1/2}(I-Q)^{1/2}D \\
D^*Q^{1/2}(I-Q)^{1/2} & D^*Q(I-Q)D
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
D^{*}(I-Q)D
\end{pmatrix},
$$

we obtain

$$
R\left(\begin{pmatrix}
Q \\
D^*Q^{1/2}(I-Q)^{1/2}
\end{pmatrix}\right) = R\left(\begin{pmatrix}
Q & Q^{1/2}(I-Q)^{1/2}D \\
D^*Q^{1/2}(I-Q)^{1/2} & D^*Q(I-Q)D
\end{pmatrix}\right)
= R(P_\mathcal{N}|_{\mathcal{H}_3 \oplus \mathcal{H}_6})
$$

by Lemma 1, which follows $Q$ is injective and positive and Lemma 4. Denote

$$
\mathcal{N}^0 = \mathcal{H}_1 \oplus \mathcal{H}_3 \oplus R\left(\begin{pmatrix}
Q \\
D^*Q^{1/2}(I-Q)^{1/2}
\end{pmatrix}\right)
$$

it is obvious that $\mathcal{N}^0 = \mathcal{N}$, so $\gamma(\mathcal{N}, \mathcal{M}) = \gamma(\mathcal{N}^0, \mathcal{M})$. For a vector $y \in \mathcal{N}^0$, there exist vectors $y_1 \in \mathcal{H}_1$, $y_3 \in \mathcal{H}_3$, and $y_5 \in \mathcal{H}_5$ such that $y = y_1 + y_3 + Qy_5 + D^*Q^{1/2}(I-Q)^{1/2}y_5$. And $P_\mathcal{M}y = y_1 + Qy_5$. Observing that $\|y\|^2 = \|y_1\|^2 + \|y_3\|^2 + \|Qy_5\|^2 + \|D^*Q^{1/2}(I-Q)^{1/2}y_5\|^2 = \|y_1\|^2 + \|y_3\|^2 + \|Qy_5\|^2$ and $\|P_\mathcal{M}y\|^2 = \|y_1\|^2 + \|Qy_5\|^2$, we have

$$
\text{dist}(y, \mathcal{M})^2 = \|y\|^2 - \|P_\mathcal{M}y\|^2 = \|y_3\|^2 + \|D^*Q^{1/2}(I-Q)^{1/2}y_5\|^2
= \|y_3\|^2 + \|Q^{1/2}(I-Q)^{1/2}y_5\|^2
$$

and

$$
\text{dist}(y, \mathcal{M} \cap \mathcal{N})^2 = \|y_3\|^2 + \|Q^{1/2}y_5\|^2.
$$

Hence

$$
\gamma(\mathcal{N}, \mathcal{M}) = \gamma(\mathcal{N}^0, \mathcal{M}) = \inf_{y \in \mathcal{N}^0, y \notin \mathcal{M}} \frac{\text{dist}(y, \mathcal{M})}{\text{dist}(y, \mathcal{M} \cap \mathcal{N})} = \inf_{y \in \mathcal{N}^0, y \notin \mathcal{M}} \frac{\sqrt{\|y_3\|^2 + \|Q^{1/2}(I-Q)^{1/2}y_5\|^2}}{\|y_3\|^2 + \|Q^{1/2}y_5\|^2} = \inf_{y \in \mathcal{N}^0, y \notin \mathcal{M}} \frac{\|Q^{1/2}(I-Q)^{1/2}y_5\|}{\|Q^{1/2}y_5\|} = \gamma(\mathcal{M}, \mathcal{N}).
$$
Here the last equation holds since $R(Q^*) = H_5$.

**Proposition 6** Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $H$ and $P_M$ and $P_N$ have the operator matrices (4). If $\mathcal{M} + \mathcal{N}$ is closed, then $\gamma(\mathcal{M}, \mathcal{N}) = (1 - \|Q\|)^{\frac{1}{2}}$.

**Proof** Note that $Q$ and $Q^*$ are injective and $D$ is unitary, then a direct computation can show $P_M + P_N$ is injective on $H_5 \ominus H_6$. Also note that $P_M + P_N$ is a positive operator in $B(H)$, then $P_M + P_N$ is dense in $H_5 \oplus H_6$. If $\mathcal{M} + \mathcal{N}$ is closed, by Lemma 4, $\mathcal{M} + \mathcal{N} = R((P_M + P_N)^\perp) = R((P_M + P_N)^\perp((P_M + P_N)^\perp)^*) = R(P_M + P_N)$ so that $P_M + P_N$ is invertible on $H_5 \oplus H_6$. Hence by formula (4), $D^*(I - Q)D$ as a restriction of $P_M + P_N$ on $H_6$ is invertible. Thus, $1 - Q$ is invertible on $H_5$ since $D$ is a unitary operator from $H_6$ onto $H_5$ and

$$\gamma(\mathcal{M}, \mathcal{N}) = \inf_{x \in \mathcal{M}, x \notin \mathcal{N}} \frac{\mathrm{dist}(x, \mathcal{N})}{\mathrm{dist}(x, \mathcal{M} \cap \mathcal{N})} = \inf_{x \in \mathcal{M}, x \notin \mathcal{N}} \frac{\sqrt{x_2^2 + (I - Q)^\perp x_5^2}}{\|x_2\|^2 + \|x_3\|^2}$$

$$= \inf_{x \in \mathcal{M}, x \notin \mathcal{N}} \frac{\|I - Q\|^\frac{1}{2} x_5}{\|x_3\|} = \frac{1}{\|I - Q\|^\frac{1}{2}} = (1 - \|Q\|)^{\frac{1}{2}}.$$

### 3 Closedness of $\mathcal{M} + \mathcal{N}$

In addition to the fundamentals of Moore-Penrose inverse, this part will present an interesting application of $\gamma(\mathcal{N}, \mathcal{M})$ and projections.

**Proposition 7** Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $H$, $P_M$ and $P_N$ have the operator matrices (4), then the following statements are equivalent:

1. $\mathcal{M} + \mathcal{N}$ is closed.
2. $\gamma(\mathcal{N}, \mathcal{M}) = (1 - \|Q\|)^{\frac{1}{2}}$ and $1 \notin \sigma(Q)$.
3. There is a constant $C > 0$ such that $\|x + y\| \geq C\|x\|$ for all $x \in \mathcal{M} \setminus (\mathcal{M} \cap \mathcal{N}), y \in \mathcal{N} \setminus (\mathcal{M} \cap \mathcal{N})$.
4. There is a constant $C > 0$ such that $\inf\{\|x - y\| : x \in \mathcal{M} \setminus (\mathcal{M} \cap \mathcal{N}), y \in \mathcal{N} \setminus (\mathcal{M} \cap \mathcal{N}), \|x\| = 1, \|y\| = 1\} \geq C$.
5. $(P_M + P_N)^\perp$ exists and $(P_M + P_N)(P_M + P_N)^\perp$ is a projection on $H \ominus (\mathcal{M}^\perp \cap \mathcal{N}^\perp)$.
6. $[P_N(I - P_M)]^\perp$ exists and $P_M + [P_N(I - P_M)]^\perp[P_N(I - P_M)]$ is a projection.
7. $[(I - P_M)P_N]^\perp$ exists and $P_M + (I - P_M)(I - P_M)P_N)^\perp$ is a projection.

**Proof** (1) $\Rightarrow$ (2) : This is immediate from Proposition 6.

(2) $\Rightarrow$ (1) : By Corollary 3, without loss of generality, assume that $\mathcal{M} \cap \mathcal{N} = \{0\}$. Now

$$\gamma(\mathcal{M}, \mathcal{N}) = \inf_{x \in \mathcal{M}} \frac{\|x + y\|}{\|x\|} \leq \frac{\|x + y\|}{\|x\|}$$

for all $x \in \mathcal{M}, y \in \mathcal{N}$. Let $C = \gamma(\mathcal{M}, \mathcal{N})$. Then $C > 0$ and $\|x + y\| \geq C\|x\|$ for all $x \in \mathcal{M} \setminus (\mathcal{M} \cap \mathcal{N}), y \in \mathcal{N} \setminus (\mathcal{M} \cap \mathcal{N})$.

(3) $\Rightarrow$ (4) : Clear.

(4) $\Rightarrow$ (1) : Without loss of generality, we may assume that $\mathcal{M} \cap \mathcal{N} = \{0\}$. Put

$$D_1 = \inf\{\|x - y\| : x \in \mathcal{M}, y \in \mathcal{N}, \|x\| = 1, \|y\| = 1\}$$

and

$$D_2 = \inf\{\|x - y\| : x \in \mathcal{M}, y \in \mathcal{N}, \|x\| = 1\}.$$

Let $\epsilon > 0, x \in \mathcal{M}$ and $\|x\| = 1$, choose $y \in \mathcal{N}$ such that $\|x - y\| \leq D_2 + \epsilon$. Since $\|x - \frac{y}{\|y\|} \leq \|x - y\| + \|\frac{y}{\|y\|}\| = \|x - y\| + \|y\| - 1 \geq \|x - y\| + \|y\| - \|x\| \leq 2\|x - \frac{y}{\|y\|}\|$. \]
\( y \parallel \leq 2D_2 + 2\epsilon, D_1 \leq \|x - \frac{y}{\|y\|}\| \leq 2D_2 + 2\epsilon, \) letting \( \epsilon \to 0 \) we see that \( D_1 \leq 2D_2 \). Hence \( \|x + y\| = \|x\| - \frac{\|y\|}{\|x\|} = \|x\| - \frac{\|y\|}{\|x\|} \geq \|x\|D_2 \geq \frac{\|y\|}{\|x\|}D_1 \geq \frac{\|y\|}{\|x\|} \) for all \( x \) in \( \mathcal{M} \) and \( y \) in \( \mathcal{N} \).

If \( x_n + y_n \to z, x_n \in \mathcal{M}, y_n \in \mathcal{N} \), then \( \|x_n - x_m\| \leq \|x_n + y_n\| + \|y_n - y_m\| = \frac{\|x\|}{\|y\|}(x_n + y_n) + (y_n - y_m) \to 0 \) since \( x_n = x \) exists and \( y_n = (x_n + y_n) - x_n \to z - x \). Since \( \mathcal{M}, \mathcal{N} \) are closed, \( x \in \mathcal{M} \), \( z - x \in \mathcal{M} \), and therefore \( z \in \mathcal{M} + \mathcal{N} \).

(1) \( \iff \) (5) : Since \( \mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N} \) has the Moore-Penrose inverse if and only if \( \mathcal{R}(\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N}) \) is closed (see [3], Theorem 2.4), it is enough to show \( \mathcal{M} + \mathcal{N} \) is closed if and only if \( \mathcal{R}(\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N}) \) is closed and \( (\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N})(\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N})^+ \) is a projection on \( \mathcal{H} \ominus (\mathcal{M}^\perp \cap \mathcal{N}^\perp) \).

If \( \mathcal{M} + \mathcal{N} \) is closed, by Proposition 6, \( \mathcal{R}(\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N}) \) is closed and \( I - Q \) is invertible on \( \mathcal{H}_5 \). Then, by Lemma 1,

\[
\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N} = 2I \oplus I \oplus I \oplus 0 \oplus \begin{pmatrix}
I + Q & Q^\frac{1}{2}(I - Q)^\frac{1}{2}D \\
0 & D^*(I - Q)D
\end{pmatrix},
\]

\[
(\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N})^+ = \frac{I}{2} \oplus I \oplus I \oplus 0 \oplus \begin{pmatrix}
I & -Q^\frac{1}{2}(I - Q)^{-\frac{1}{2}}D \\
-0 & D^*(I - Q)^{-\frac{1}{2}}D
\end{pmatrix}
\]

exists, and \( (\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N})(\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N})^+ = I \oplus I \oplus 0 \oplus I \oplus I \) is a projection on \( \mathcal{H} \ominus (\mathcal{M}^\perp \cap \mathcal{N}^\perp) \).

On the other hand, if \( \mathcal{R}(\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N}) \) is closed, with the notation of Lemma 4,

\[
\mathcal{R}(\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N}) \subseteq \mathcal{R}((\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N})^+) = \mathcal{M} + \mathcal{N}
\]

and

\[
\mathcal{R}(\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N}) = \mathcal{R}((\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N})^+) = \mathcal{M} + \mathcal{N},
\]

therefore \( \mathcal{M} + \mathcal{N} \) is closed.

(5) \( \implies \) (6) : From (4), we have

\[
\mathcal{P} \mathcal{N}(I - \mathcal{P} \mathcal{M}) = 0 \oplus 0 \oplus I \oplus 0 \oplus \begin{pmatrix}
0 & Q^\frac{1}{2}(I - Q)^\frac{1}{2}D \\
0 & D^*(I - Q)D
\end{pmatrix}
\]

\[
(\mathcal{P} \mathcal{N}(I - \mathcal{P} \mathcal{M}))^* = \begin{pmatrix} 0 & 0 \\ 0 & D^*(I - Q)D \end{pmatrix}.
\]

If \( (\mathcal{P} \mathcal{M} + \mathcal{P} \mathcal{N})^+ \) exists, then \( I - Q \) invertible. Therefore

\[
[\mathcal{P} \mathcal{N}(I - \mathcal{P} \mathcal{M})]^*[\mathcal{P} \mathcal{N}(I - \mathcal{P} \mathcal{M})]^+ = 0 \oplus 0 \oplus I \oplus 0 \oplus \begin{pmatrix}
0 & 0 \\
0 & D^*(I - Q)^{-1}D
\end{pmatrix}
\]

From (1), we have

\[
[\mathcal{P} \mathcal{N}(I - \mathcal{P} \mathcal{M})]^+ = ([\mathcal{P} \mathcal{N}(I - \mathcal{P} \mathcal{M})]^*[\mathcal{P} \mathcal{N}(I - \mathcal{P} \mathcal{M})]^+][\mathcal{P} \mathcal{N}(I - \mathcal{P} \mathcal{M})]^*
\]

\[
= 0 \oplus 0 \oplus I \oplus 0 \oplus 0 \oplus I.
\]

exists and

\[
\mathcal{P} \mathcal{M} + [\mathcal{P} \mathcal{N}(I - \mathcal{P} \mathcal{M})]^+[\mathcal{P} \mathcal{N}(I - \mathcal{P} \mathcal{M})] = I \oplus I \oplus I \oplus 0 \oplus I \oplus I
\]
is a projection.

\[(6) \Rightarrow (7) : \text{Since } [(I - P_M)P_N]^\bot = [(P_N(I - P_M))^\bot]^\bot, \text{this implies} \]
\[(I - P_M)P_N]^\bot = 0 \oplus 0 \oplus I \oplus 0 \oplus 0 \oplus I \text{ exists and } P_M + (I - P_M)[(I - P_M)P_N]^\bot = I \oplus I \oplus 0 \oplus I \oplus I \]
\text{is a projection.}

\[(7) \Rightarrow (5) : \text{If } [(I - P_M)P_N]^\bot \text{ exists, then it is easy to see that} I - Q \text{ is invertible. Hence} \]
\[(P_M + P_N)^\bot = \frac{1}{2} \oplus I \oplus 0 \oplus \begin{pmatrix}
I & -Q^\frac{1}{2}(I - Q)^{-\frac{1}{2}}D
\end{pmatrix}
\]
\text{exists, and} \]
\[(P_M + P_N)(P_M + P_N)^\bot = I \oplus I \oplus 0 \oplus I \oplus I \text{ is a projection on } H \oplus (M^\perp \cap N^\perp). \]

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