ON PROPERTIES OF MEROMORPHIC SOLUTIONS FOR COMPLEX DIFFERENCE EQUATION OF Malmquist Type*

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Abstract In this paper, we present the properties on zeros, fixed points, poles, Borel exceptional value of finite order transcendental meromorphic solutions of complex difference equation of Malmquist type

$$\sum_{j=1}^{n} f(z+c_j) = R(f(z)) = \frac{P(f(z))}{Q(f(z))} = a_p f(z)^p + a_{p-1} f(z)^{p-1} + \cdots + a_1 f(z) + a_0$$

where $n(\in \mathbb{N}) \geq 2$, $P(f(z))$ and $Q(f(z))$ are relatively prime polynomials in $f(z)$ with rational coefficients $a_s$ ($s = 0, 1, \cdots, p$) and $b_t$ ($t = 0, 1, \cdots, q$) such that $a_0 a_p b_q \neq 0$, and also consider the existence and the forms on rational solutions of this type of difference equations. Some examples are also listed to show that the assumptions of theorems, in certain senses, are the best possible.

Key words zeros; poles; fixed-points; Borel exceptional value; difference equation

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1 Introduction and Main Results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notions of Nevanlinna’s value distribution theory of meromorphic functions (see, e.g., [6, 16]). In addition, we use $\sigma(f)$ and $\lambda(f)$ to denote the order and the exponent of convergence of the zero sequences of a meromorphic function $f(z)$. We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of $f(z)$ as follows.

$$\tau(f) = \lim_{r \to \infty} \frac{\log N(r, \frac{1}{f(z)})}{\log r}.$$
We specify the notion of small functions as follows. Given a meromorphic function $f(z)$, the family of all meromorphic functions $a(z)$ such that $T(r,a) = S(r,f)$ is denoted by $S$. For convenience, we also include all constants in $S$.

As we all know, the celebrated Malmquist theorem showed that a complex differential equation of the type

$$f'(z) = R(z, f(z)),$$

(1.1)

where $R(z, f(z))$ is rational in both arguments, and which admits a transcendental meromorphic solution $f(z)$ in the complex plane, reduces into a Riccati differential equation

$$f'(z) = a(z) + b(z)f(z) + c(z)f(z)^2$$

(1.2)

with rational coefficients. For more details concerning equations (1.1) and (1.2) as well as for generations of the Malmquist theorem, see, e.g. [9].

After that, a number of results on difference equations arising from reasoning of Malmquist type (see, e.g. [1, 4, 7, 8, 12, 14]), in which the growth of order and existence are considered.

Recently, Chen and Shon considered the value distribution of finite order transcendental meromorphic solutions, and the forms of rational solutions of difference Painlevé II equation. We now recall their results as follows.

**Theorem 1.A** ([3, Theorem 1]) Let $a, b, c$ be constants with $ac \neq 0$. If $f(z)$ is a finite order transcendental meromorphic solution of the difference Painlevé II equation

$$f(z + 1) + f(z - 1) = \frac{(az + b)f(z) + c}{1 - f(z)^2},$$

(1.3)

then

(i) $f(z)$ has at most one nonzero finite Borel exceptional value;

(ii) $\lambda(\frac{1}{f}) = \lambda(f) = \sigma(f)$;

(iii) $f(z)$ has infinitely many fixed points and satisfies $\tau(f) = \sigma(f)$.

**Theorem 1.B** ([3, Theorem 2]) Let $a, b, c$ be constants with $ac \neq 0$. Suppose that a rational function

$$f(z) = \frac{P(z)}{Q(z)} = \frac{pz^m + p_{m-1}z^{m-1} + \cdots + p_0}{qz^n + q_{n-1}z^{n-1} + \cdots + q_0}$$

is a solution of equation (1.3), where $P(z)$ and $Q(z)$ are relatively prime polynomials, $p, p_{m-1}, \cdots, p_0$ and $q, q_{n-1}, \cdots, q_0$ are constants with $pq \neq 0$. Then

$$n = m + 1, \quad p = -c/a q_1.$$

A natural question is whether the similar results hold for more general difference equations of Malmquist type. Here, we answer this question and obtain the following results.

**Theorem 1.1** Suppose that $c_1, c_2, \cdots, c_n$ are distinct, nonzero constants. If complex difference equation of Malmquist type

$$\sum_{j=1}^{n} f(z + c_j) = R(f(z)) = \frac{P(f(z))}{Q(f(z))} = \frac{a_pf(z)^p + a_{p-1}f(z)^{p-1} + \cdots + a_1f(z) + a_0}{b_qf(z)^q + b_{q-1}f(z)^{q-1} + \cdots + b_1f(z) + b_0}$$

(1.4)

admits a finite order transcendental meromorphic solution $f(z)$, where $n (\in \mathbb{N}) \geq 2$, $P(f(z))$ and $Q(f(z))$ are relatively prime polynomials in $f(z)$ with rational coefficients $a_s$ $(s = 0, 1, \cdots, p)$ and $b_t$ $(t = 0, 1, \cdots, q)$ such that $a_0a_pb_q \neq 0$. Then
(1) \( f(z) \) has infinitely many zeros and satisfies \( \lambda(f) = \sigma(f) \);
(2) \( f(z) \) has infinitely many fixed points and satisfies \( \tau(f) = \sigma(f) \);
(3) \( f(z) \) has infinitely many poles and satisfies \( \lambda \left( \frac{1}{f} \right) = \sigma(f) \);
(4) \( f(z) \) has no Borel exceptional value.

**Theorem 1.2** Suppose that \( c_1, c_2, \ldots, c_n \) are distinct, nonzero constants. Consider complex difference equation (1.4) of Malmquist type

\[
\sum_{j=1}^{n} f(z + c_j) = R(f(z)) = \frac{P(f(z))}{Q(f(z))} = \frac{a_0 f(z)^p + a_{p-1} f(z)^{p-1} + \cdots + a_1 f(z) + a_0}{b_q f(z)^q + b_{q-1} f(z)^{q-1} + \cdots + b_1 f(z) + b_0},
\]

where \( n \in \mathbb{N} \geq 2 \), \( P(f(z)) \) and \( Q(f(z)) \) are relatively prime polynomials in \( f(z) \) with constant coefficients \( a_s \) \((s = 0, 1, \ldots, p)\) and \( b_t \) \((t = 0, 1, \ldots, q)\) such that \( a_0 a_p b_q \neq 0 \). Then

1. Equation (1.4) has nonconstant rational solution which has the form

\[
f(z) = \frac{P_0(z)}{Q_0(z)} + A,
\]

where \( A \) are nonzero constant, \( P_0(z) \) and \( Q_0(z) \) are relatively prime polynomials in \( z \) with \( \deg \{ P_0 \} < \deg \{ Q_0 \} \).

2. Equation (1.4) must have a nonzero constant solution \( C \in \mathbb{C} \) satisfying

\[
P(C) - nCQ(C) = 0.
\]

**Example 1.1** Let \( c_j \in \mathbb{C} \) \((j = 1, 2)\) be a constant such that \( c_j \neq \frac{\pi}{2} m \) \((m \in \mathbb{Z}, \ j = 1, 2)\), and \( C_j = -\tan c_j \neq 0, \infty \) \((j = 1, 2)\). Then complex difference equation of Malmquist type

\[
f(z + 2) + f(z + 1) = \frac{(C_1 + C_2)f^2(z) + 2(1 - C_1 C_2)f(z) - (C_1 + C_2)}{C_1 C_2 f^2(z) + (C_1 + C_2)f(z) + 1}
\]

is solved by \( f(z) = \tan z \) satisfying \( \lambda(\frac{1}{f}) = \lambda(f) = \tau(f) = \sigma(f) = 1 \).

**Example 1.2** Suppose that \( A \) is a nonzero constant. Then complex difference equation of Malmquist type

\[
f(z + 1) + f(z - 1) = \frac{-2Af^2(z) + 2(2A^2 + 1)f(z) - 2A^3}{-f^2(z) + 2Af(z) + 1 - A^2}
\]

has a rational solution \( f_1(z) = \frac{1}{z+1} + A \) and a constant solution \( f_2(z) = A \) satisfying (1.5) and (1.6), respectively.

The following example shows that the restriction in Theorem 1.1 to \( a_0 \neq 0 \) is essential.

**Example 1.3** Complex difference equation of Malmquist type

\[
f(z + 1) + f(z - 1) = \frac{2f(z)}{-f^2(z) + 1}
\]

has a transcendental meromorphic solution \( f(z) = \frac{1}{e^{\pi i z + \pi i}} \). But

\[
\tau(f) = \lambda \left( \frac{1}{f} \right) = \sigma(f) = 1 \quad \text{and} \quad \lambda(f) = 0.
\]
2 Lemmas for the Proofs

In order to prove theorems, we need some preliminaries.

**Lemma 2.1** ([5, Theorem 3.2, 10, Theorem 2.4]) Let \( f(z) \) be non-constant finite order meromorphic solution of
\[
P(z, f(z)) = 0,
\]
where \( P(z, f(z)) \) is difference polynomials in \( f(z) \). If \( P(z, a) \neq 0 \) for \( a \in S \), then
\[
m \left( r, \frac{1}{f-a} \right) = S(r, f)
\]
for all \( r \) outside of a possible exceptional set with finite logarithmic measure.

**Proof** Suppose that \( f(z) \) is a transcendental meromorphic of finite order to equation (1.4). We deduce from (1.4) that
\[
P_1(z, f(z)) = [b_q f(z)^q + b_{q-1} f(z)^{q-1} + \cdots + b_1 f(z) + b_0] \sum_{j=1}^{n} f(z + c_j) - [a_p f(z)^p + a_{p-1} f(z)^{p-1} + \cdots + a_1 f(z) + a_0] \equiv 0.
\]
We notice that
\[
P_1(z, 0) = -a_0 \neq 0.
\]
It follows from Lemma 2.1 that
\[
m \left( r, \frac{1}{f(z)} \right) = S(r, f)
\]
for all \( r \) outside of a possible exceptional set with finite logarithmic measure. Therefore,
\[
N \left( r, \frac{1}{f(z)} \right) = T(r, f) + S(r, f)
\]
for all \( r \) outside of a possible exceptional set of finite logarithmic measure. Thus, \( \lambda(f) = \sigma(f) \).

The proof of Lemma 2.2 is completed. \( \square \)

**Lemma 2.3** Let \( a_s(z) (s = 1, 2, \cdots, p) \) and \( b_t(z) (t = 1, 2, \cdots, q) \) be rational functions such that \( a_0 b_q \neq 0 \). If equation (1.4) admits a finite order transcendental meromorphic solution \( f(z) \), then \( f(z) \) has infinitely many zeros and satisfies \( \lambda(f) = \sigma(f) \).

**Proof** Suppose that \( f(z) \) is a transcendental meromorphic of finite order to equation (1.4). Set \( g(z) = f(z) - z \), then \( \sigma(g) = \sigma(f) < +\infty \). Substituting \( f(z) = g(z) + z \) into (1.4), we conclude that
\[
P_2(z, g(z)) = \{b_q [g(z) + z]^q + b_{q-1} [g(z) + z]^{q-1} + \cdots + b_1 [g(z) + z] + b_0\}
\]
\[
\times \sum_{j=1}^{n} [g(z + c_j) + nz + c_n + c_{n-1} + \cdots + c_1]
\]
\[
- \{a_p [g(z) + z]^p + a_{p-1} [g(z) + z]^{p-1} + \cdots + a_1 [g(z) + z] + a_0\} \equiv 0.
\]
Since \( P(f(z)) \) and \( Q(f(z)) \) are relatively prime polynomials in \( f(z) \), \( P(z) \) and \( Q(z) \) are also relatively prime polynomials in \( z \). Thus we get

\[
P_2(z, 0) = [b_0 z^q + b_{q-1} z^{q-1} + \cdots + b_1 z + b_0] \cdot [n z + c_n + c_{n-1} + \cdots + c_1] \\
- [a_p z^p + a_{p-1} z^{p-1} + \cdots + a_1 z + a_0] \\
= [n z + c_n + c_{n-1} + \cdots + c_1] Q(z) - P(z) \neq 0.
\]

It follows from Lemma 2.1 that

\[
m \left( r, \frac{1}{g(z)} \right) = S(r, f)
\]

for all \( r \) outside of a possible exceptional set with finite logarithmic measure. Therefore,

\[
N \left( r, \frac{1}{f(z) - z} \right) = N \left( r, \frac{1}{g(z)} \right) = T(r, g) + S(r, g) = T(r, f) + S(r, f)
\]

for all \( r \) outside of a possible exceptional set of finite logarithmic measure. So,

\[
\tau(f) = \lambda(g) = \sigma(g) = \sigma(f).
\]

The proof of Lemma 2.3 is completed. \( \square \)

**Lemma 2.4** ([7, Proposition 8]) Let \( c_1, c_2, \cdots, c_n \in \mathbb{C} \setminus \{0\} \). If the difference equation

\[
\sum_{i=1}^{n} f(z + c_i) = R(z, f(z)) = \frac{a_0(z) + a_1(z) f(z) + \cdots + a_p(z) f(z)^p}{b_0(z) + b_1(z) f(z) + \cdots + b_q(z) f(z)^q}
\]

with rational coefficients \( a_s(z) \) \((s = 1, 2, \cdots, p)\) and \( b_t(z) \) \((t = 1, 2, \cdots, q)\) admits a transcendental meromorphic solution of finite order, then \( d = \max\{p, q\} \leq n \).

**Lemma 2.5** ([10, Theorem 2.3]) Let \( f(z) \) be a transcendental meromorphic solution of finite order \( \sigma \) of a difference equation of the form

\[
U(z, f) P(z, f) = Q(z, f),
\]

where \( U(z, f), P(z, f) \) and \( Q(z, f) \) are difference polynomials with all the coefficients \( \alpha_\lambda(z) \) are small functions as understood in the usual Nevanlinna theory, i.e., \( T(r, \alpha_\lambda) = O(r^{\sigma - 1 + \varepsilon}) + S(r, f) \). The maximum total degree \( \deg_f U(z, f) = n \) in \( f(z) \) and its shifts, and \( \deg_f Q(z, f) \leq n \). Moreover, we assume that \( U(z, f) \) contains just one term of maximal total degree in \( f(z) \) and its shifts. Then for each \( \varepsilon > 0 \),

\[
m(r, P(z, f)) = O(r^{\sigma - 1 + \varepsilon}) + S(r, f),
\]

possibly outside of an exceptional set of finite logarithmic measure.

**Lemma 2.6** (Valiron-Mohon’ko [9, 11, 13]) Let \( f(z) \) be a meromorphic function. Then for all irreducible rational functions in \( f(z) \),

\[
R(z, f(z)) = \frac{a_0(z) + a_1(z) f(z) + \cdots + a_p(z) f(z)^p}{b_0(z) + b_1(z) f(z) + \cdots + b_q(z) f(z)^q}
\]

with meromorphic coefficients \( a_i(z) \) \((i = 0, 1, \cdots, p)\) and \( b_j(z) \) \((j = 0, 1, \cdots, q)\), the characteristic function of \( R(z, f(z)) \) satisfies

\[
T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),
\]
where \( d = \max\{p, q\} \) and
\[
\Psi(r) = \max_{i,j} \{T(r, a_i), T(r, b_j)\}.
\]

In the particular case when
\[
\begin{cases}
  T(r, a_i) = S(r, f) & i = 0, 1, \ldots, p, \\
  T(r, b_j) = S(r, f) & j = 0, 1, \ldots, q,
\end{cases}
\]
we have
\[
T(R(z, f(z))) = dT(r, f(z)) + S(r, f).
\]

**Lemma 2.7** ([2, Theorem 2.2]) Let \( f(z) \) be a meromorphic function with exponent of convergence of poles \( \lambda(\frac{1}{f}) = \lambda < +\infty, \eta \neq 0 \) be fixed, then for each \( \varepsilon > 0 \),
\[
N(r, f(z + \eta)) = N(r, f) + O(r^{\lambda-1+\varepsilon}) + O(\log r).
\]

**Lemma 2.8** ([2, Corollary 2.6]) Let \( \eta_1 \) and \( \eta_2 \) be two complex numbers such that \( \eta_1 \neq \eta_2 \) and let \( f(z) \) be a finite order meromorphic function. Let \( \sigma \) be the order of \( f(z) \), then for each \( \varepsilon > 0 \), we have
\[
m \left( r, \frac{f(z + \eta_1)}{f(z + \eta_2)} \right) = O(r^{\sigma-1+\varepsilon}).
\]

**Lemma 2.9** Let \( a_s(z) (s = 1, 2, \ldots, p) \) and \( b_t(z) (t = 1, 2, \ldots, q) \) be rational functions such that \( a_p b_q \neq 0 \). If equation (1.4) admits a finite order transcendental meromorphic solution \( f(z) \), then \( f(z) \) has infinitely many poles and satisfies \( \lambda(\frac{1}{f}) = \sigma(f) \).

**Proof** Suppose that \( f(z) \) is a transcendental meromorphic solution of finite order to equation (1.4). By applying Lemma 2.4 to equation (1.4), we obtain \( 1 \leq p, q \leq \max\{p, q\} = d \leq n \). Set \( \sigma(f) = \sigma \) and
\[
H(f(z)) = \sum_{j=1}^{n} f(z + c_j).
\]
If \( 1 \leq p \leq q = d \leq n \), then (1.4) can be written as
\[
Q(f(z))H(f(z)) = P(f(z)).
\]
It follows from Lemma 2.5 that
\[
m(r, H(f(z))) = O(r^{\sigma-1+\varepsilon}) + S(r, f). \tag{2.1}
\]
We now apply Lemma 2.6 to equation (1.4) and conclude that
\[
T(r, H(f(z))) = T \left( r, \frac{P(f(z))}{Q(f(z))} \right) = QT(r, f) + S(r, f). \tag{2.2}
\]
It follows from Lemma 2.7, (2.1) and (2.2) that
\[
nN(r, f) \geq N(r, H(f(z))) + O(r^{\lambda-1+\varepsilon}) + O(\log r) = qT(r, f) + O(r^{\sigma-1+\varepsilon}) + S(r, f).
\]
Thus, \( \lambda(\frac{1}{f}) = \sigma(f) \).
If $1 \leq q < p = d \leq n$, then by applying Lemma 2.6 to equation (1.4), we get
\[
T(r, H(f(z))) = T\left(r, \frac{P(f(z))}{Q(f(z))}\right) = pT(r, f) + S(r, f).
\] (2.3)

However, we deduce from Lemma 2.8 and (1.4) that
\[
m(r, H(f(z))) = m\left(r, \frac{H(f(z))}{f(z)} \cdot f(z)\right) \leq m\left(r, \frac{H(f(z))}{f(z)}\right) + m(r, f(z))
\]
\[
= m\left(r, \sum_{j=1}^{n} \frac{f(z + c_j)}{f(z)}\right) + m(r, f(z)) \leq T(r, f) + O(r^{\sigma - 1 + \varepsilon}).
\] (2.4)

It follows from Lemma 2.7, (2.3) and (2.4) that
\[
nN(r, f) \geq N(r, H(f(z))) + O(r^{\lambda - 1 + \varepsilon}) + O(\log r)
\]
\[
= T(r, H(f(z))) - m(r, H(f(z))) + O(r^{\lambda - 1 + \varepsilon}) + O(\log r)
\]
\[
\geq (p - 1)T(r, f) + O(r^\sigma + \varepsilon) + O(\log r).
\]

Thus, $\lambda(\frac{1}{f}) = \sigma(f)$. The proof of Lemma 2.9 is completed. \hfill \Box

**Lemma 2.10** ([15, Theorem 1.51]) Suppose that $f_j(z)$ ($j = 1, 2, \cdots, n, n \geq 2$) are meromorphic functions, $g_j(z)$ ($j = 1, 2, \cdots, n$) are entire functions satisfying the following conditions:

1. $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} = 0$.

2. $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$.

3. For $1 \leq j \leq n$, $1 \leq h < k \leq n$,
\[
T(r, f_j) = o\left\{T(r, e^{g_h - g_k})\right\} (r \to +\infty, r \notin E),
\]

where $E \subset (1, +\infty)$ is of finite linear measure or finite logarithmic measure.

Then $f_j(z) \equiv 0$ ($j = 1, 2, \cdots, n$).

### 3 Proof of Theorem 1.1

By Lemma 2.2, Lemma 2.3 and Lemma 2.9, we see that (1)–(3) of Theorem 1.1 hold. So we only prove (4) of Theorem 1.1. Suppose that $f(z)$ is a transcendental meromorphic solution of finite order to equation (1.4). First, we deduce from Lemma 2.2 and Lemma 2.9 that 0 and $\infty$ are not Borel exceptional values. If $\beta \in \mathbb{C} \setminus \{0\}$ is a Borel exceptional value, then $\lambda(f(z) - \beta) < \sigma(f(z) - \beta) = \sigma(f) < +\infty$. Thus, there exists a $m \in \mathbb{N}$ such that $\sigma(f(z) - \beta) = \sigma(f) = m < +\infty$, and $f(z) - \beta$ can be written as
\[
f(z) - \beta = h(z)e^{\varepsilon z^m},
\] (3.1)

where $c$ is a nonzero constant, $h(z)$ is a meromorphic function satisfying
\[
\sigma(h) < \sigma(f) = m.
\] (3.2)

Therefore,
\[
f(z + c_j) = \beta + h(z + c_j)h_j(z)e^{\varepsilon z^m} \quad (j = 1, 2, \cdots, n),
\] (3.3)
where
\[ h_j(z) = \exp \left( c \sum_{k=1}^{m} \binom{m}{k} z^{m-k} c_j^k \right) \quad \text{and} \quad \sigma(h_j) = m - 1 \quad (j = 1, 2, \cdots, n). \] (3.4)

Substituting (3.1) and (3.3) into equation (1.4), we obtain
\[
\begin{align*}
& b_q \left( \sum_{j=1}^{n} h(z+c_j) h_j(z) \right) h(z)^q e^{(q+1)cz^{m}} + B_q(z)e^{cz^{m}} + \cdots + B_1(z)e^{cz^{m}} + n\beta Q(\beta) \\
& = a_p h(z)^p e^{cxz^{m}} + A_{p-1}(z)e^{(p-1)cxz^{m}} + \cdots + A_1(z)e^{cxz^{m}} + P(\beta),
\end{align*}
\] (3.5)
where \( P(\beta), Q(\beta) \) are the values of \( P(f(z)) \) and \( Q(f(z)) \) at \( f(z) = \beta \) respectively, \( A_s(z) \) (\( s = 1, 2, \cdots, p - 1 \)) and \( B_t(z) \) (\( t = 1, 2, \cdots, q \)) are polynomials in \( h(z) \) and its shifts with \( \sigma(A_s) \leq m - 1 \quad (s = 1, 2, \cdots, p - 1) \) and \( \sigma(B_t) \leq m - 1 \quad (t = 1, 2, \cdots, q) \).

By rearranging (3.5), it follows from Lemma 2.10 that
\[
\begin{align*}
& b_q \left( \sum_{j=1}^{n} h(z+c_j) h_j(z) \right) h(z)^q \equiv 0, \quad p < q + 1, \\
& b_q \left( \sum_{j=1}^{n} h(z+c_j) h_j(z) \right) h(z)^q - a_p h^p(z) \equiv 0, \quad p = q + 1, \\
& a_p h^p(z) \equiv 0, \quad p > q + 1.
\end{align*}
\]
These are impossible since \( a_0a_p b_q \not\equiv 0 \), \( h(z), h(z+c_j) \) and \( h_j(z) \) (\( j = 1, 2, \cdots, n \)) satisfy (3.2)-(3.4). Hence \( \lambda(f(z) - \beta) = \sigma(f) \). The proof of Theorem 1.1 is completed. \( \square \)

4 Proof of Theorem 1.2

(1) Suppose that \( f(z) \) is a rational solution of equation (1.4) and \( f(z) \) has poles \( z_1, z_2, \cdots, z_k \) with multiplicities \( \lambda_1, \lambda_2, \cdots, \lambda_k \) respectively. So \( f(z) \) can be rewritten as the sum of fraction in lowest terms and polynomial as follows.
\[
f(z) = \sum_{j=1}^{k} \left[ \frac{c_j \lambda_j}{(z-z_j)^{\lambda_j}} + \frac{c_j \lambda_{j-1}}{(z-z_j)^{\lambda_j-1}} + \cdots + \frac{c_j 1}{z-z_j} \right] + A_0 + A_1 z + \cdots + A_\nu z^\nu, \]
(4.1)
where \( c_j \lambda_j \not\equiv 0 \), \( c_j \lambda_{j-1}, \cdots, c_j 1 \) (\( j = 1, 2, \cdots, k \)) and \( A_\mu \) (\( \mu = 0, 1, 2, \cdots, \nu \)) are constants.

First, we affirm that \( A_\mu \equiv 0 \) (\( \mu = 1, 2, \cdots, \nu \)). If \( A_\nu \) \( \not\equiv 0 \), it follows from (4.1) that, for all sufficiently large \( z \),
\[
\begin{align*}
& f(z) = A_\nu z^\nu(1 + o(1)), \\
& f(z+c_j) = A_\nu(z+c_j)^\nu(1 + o(1)) = A_\nu z^\nu(1 + o(1)) \quad (j = 1, 2, \cdots, n).
\end{align*}
\] (4.2)
Substituting (4.2) into (1.4), we obtain
\[
\begin{align*}
& n A_\nu z^\nu(1 + o(1)) \\
& = a_p[A_\nu z^\nu(1 + o(1))]^p + a_{p-1}[A_\nu z^\nu(1 + o(1))]^{p-1} + \cdots + a_1[A_\nu z^\nu(1 + o(1))] + a_0 \\
& = b_q[A_\nu z^\nu(1 + o(1))]^q + b_{q-1}[A_\nu z^\nu(1 + o(1))]^{q-1} + \cdots + b_1[A_\nu z^\nu(1 + o(1))] + b_0.
\end{align*}
\]
Therefore,
\[ nb_q A_{\nu}^{p+1} z^{(q+1)\nu}(1+o(1)) + nb_{q-1} A_{\nu}^{p} z^{(p-1)\nu}(1+o(1)) + \cdots + nb_{0} A_{\nu} z^{\nu}(1+o(1)) \]
\[ = a_p A_{\nu}^{p} z^{\nu}(1+o(1)) + a_{p-1} A_{\nu}^{p-1} z^{(p-1)\nu}(1+o(1)) + \cdots + a_{1} A_{\nu} z^{\nu}(1+o(1)) + a_{0}. \]  
(4.3)

If \( p \neq q + 1 \), (4.3) is a contradiction for all sufficiently large \( z \).

If \( p = q + 1 \), (4.3) turns into
\[ (a_p - nb_{p-1}) A_{\nu}^{p} z^{\nu}(1+o(1)) + (a_{p-1} - nb_{p-2}) A_{\nu}^{p-1} z^{(p-1)\nu}(1+o(1)) \]
\[ + \cdots + (a_1 - nb_{0}) A_{\nu} z^{\nu}(1+o(1)) + a_{0} = 0. \]  
(4.4)

Comparing the coefficients of both sides of (4.4), we deduce that
\[ a_j - nb_{j-1} = 0 \quad (j = 1, 2, \cdots, p) \quad \text{and} \quad a_0 = 0. \]

From these, we conclude that \( P(f(z)) = n f(z)Q(f(z)) \). This is impossible since \( P(f(z)) \) and \( Q(f(z)) \) are relatively prime. Similarly, \( A_\mu = 0 \) (\( \mu = 1, 2, \cdots, \nu - 1 \)).

Second, we assume \( A_0 \neq 0 \). If \( A_0 = 0 \), then, for all sufficiently large \( z \),
\[ f(z) \to 0 \quad \text{and} \quad f(z + c_j) \to 0 \quad (j = 1, 2, \cdots, n). \]  
(4.5)

Substituting (4.5) into (1.4), we obtain, for all sufficiently large \( z \),
\[ \frac{a_0}{b_0} \to 0. \]

Thus, \( a_0 = 0 \). This is a contradiction. So, by (4.1), \( f(z) \) can be rewritten as the form
\[ f(z) = \frac{P_0(z)}{Q_0(z)} + A_0, \]
where \( P_0(z) \) and \( Q_0(z) \) are relatively prime polynomials in \( z \) with
\[ \deg P_0(z) < \deg Q_0(z). \]

(2) This result is trivial. Here we give a simple proof. By the fundamental theorem of algebra, we know that the equation
\[ P(\zeta) - n\zeta Q(\zeta) = 0 \]  
(4.6)

must have a nonzero constant solution \( C \in \mathbb{C} \) since \( a_0 \neq 0 \). This shows that equation (1.4) exist a nonzero constant solution \( C \) which satisfies \( P(C) - nCQ(C) = 0 \). \( \square \)

5 Appendix: the Difference Analogue of Valiron-Mohon’ko Theorem

Valiron-Mohon’ko theorem showed that the relation between the characteristic function of \( f(z) \) and the characteristic function of irreducible rational functions
\[ R(z, f(z)) = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \cdots + b_q(z)f(z)^q} \]
with meromorphic coefficients \( a_i(z) \in \mathcal{S} \ (i = 0, 1, \cdots, p) \) and \( b_j(z) \in \mathcal{S} \ (j = 0, 1, \cdots, q) \).

A nature question is whether the Valiron-Mohon’ko theorem could hold for any rational function of \( f(z) \) and its shifts \( f(z + c) \).

**Example 5.1** Set \( f(z) = e^{z\log 2} - 1 \) and

\[
R(z, f(z)) = \frac{f(z + 1) - f(z) - 1}{f(z)^n}, \quad n = 2, 3, \cdots.
\]

Then \( T(r, R(z, f(z))) = (n - 1)T(r, f) + S(r, f) \).

**Example 5.2** Set \( f(z) = \tan z \) and

\[
R(z, f(z)) = \frac{f(z + \frac{\pi}{2}) - f(z) - 1}{f(z)^n}, \quad n = 2, 3, \cdots.
\]

Then \( T(r, R(z, f(z))) = (n + 1)T(r, f) + S(r, f) \).

These two examples show that Valiron-Mohon’ko theorem isn’t true for any rational function of \( f(z) \) and its shifts \( f(z) \). But we will prove that the difference analogue of Valiron-Mohon’ko theorem is true for finite order transcendental meromorphic function \( f(z) \) under certain assumption as follows.

**Theorem 5.1** Let \( f(z) \) is a finite order transcendental meromorphic function with

\[
N(r, f(z)) + N\left(r, \frac{1}{f(z)}\right) = S(r, f).
\]

(5.1)

\( S(z, f(z)) \) and \( K(z, f(z)) \) are difference polynomials in \( f(z) \) and its shifts \( f(z + c) \) with small functions of \( f(z) \) as its coefficients. Moreover, we assume that \( S(z, f(z)) \) and \( K(z, f(z)) \) are relatively prime polynomials in \( f(z) \), and contain just one term of maximal total degree in \( f(z) \) and its shifts, respectively. Then the characteristic function of

\[
R(z, f(z)) = \frac{S(z, f(z))}{K(z, f(z))} = \frac{\sum_{\lambda \in I} a_{\lambda}(z) \prod_{i=1}^{\sigma_{\lambda}} f(z + \alpha_{\lambda,i})^{l_{\lambda,i}}}{\sum_{\mu \in J} b_{\mu}(z) \prod_{j=1}^{\sigma_{\mu}} f(z + \beta_{\mu,j})^{m_{\mu,j}}}
\]

satisfies

\[
T(r, R(z, f(z))) = \max\{p, q\}T(r, f(z)) + S(r, f),
\]

where \( I \) and \( J \) are finite sets of multi-index respectively, \( \alpha_{\lambda,i}, \beta_{\mu,j} \in \mathbb{C} \), the coefficients \( a_{\lambda}(z), b_{\mu}(z) \in \mathcal{S}, p = \deg_f(S(z, f(z))) \) and \( q = \deg_f(K(z, f(z))) \).

**Proof** Set

\[
\Delta_k := \left\{ \lambda \in I : \sum_{i=1}^{\sigma_{\lambda}} l_{\lambda,i} = k \right\}, \quad k = 0, 1, \cdots, p.
\]

Now we rearrange the expression of the difference polynomial \( S(z, f(z)) \) by collecting together all terms having the same total degree and obtain

\[
S(z, f(z)) = \sum_{\lambda \in I} a_{\lambda}(z) \prod_{i=1}^{\sigma_{\lambda}} f(z + \alpha_{\lambda,i})^{l_{\lambda,i}}
\]

\[
= \sum_{\lambda \in \Delta_p} a_{\lambda}(z) \prod_{i=1}^{\sigma_{\lambda}} f(z + \alpha_{\lambda,i})^{l_{\lambda,i}} + \sum_{\lambda \in \Delta_{p-1}} a_{\lambda}(z) \prod_{i=1}^{\sigma_{\lambda}} f(z + \alpha_{\lambda,i})^{l_{\lambda,i}} + \cdots
\]

\[
+ \sum_{\lambda \in \Delta_1} a_{\lambda}(z) \prod_{i=1}^{\sigma_{\lambda}} f(z + \alpha_{\lambda,i})^{l_{\lambda,i}} + \sum_{\lambda \in \Delta_0} a_{\lambda}(z).
\]
Thus, we can rewrite $S(z, f(z))$ as the form

$$S(z, f(z)) = \tilde{a}_p(z) f(z)^p + \tilde{a}_{p-1}(z) f(z)^{p-1} + \cdots + \tilde{a}_1(z) f(z) + \tilde{a}_0(z), \quad (5.2)$$

where

$$\tilde{a}_0(z) = \sum_{\lambda \in \Delta_0} a_\lambda(z) \quad \text{and} \quad \tilde{a}_k(z) = \sum_{\lambda \in \Delta_k} a_\lambda(z) \prod_{i=1}^{\sigma_\lambda} \left( \frac{f(z + \alpha_{\lambda,i})}{f(z)} \right)^{l_{\lambda,i}}, \quad k = 1, \ldots, p. \quad (5.3)$$

Obviously, $\tilde{a}_p(z) = a_\lambda(z) \prod_{i=1}^{\sigma_\lambda} \left( \frac{f(z + \alpha_{\lambda,i})}{f(z)} \right)^{l_{\lambda,i}} \neq 0$ and $\sum_{i=1}^{\sigma_\lambda} l_{\lambda,i} = p$ since $S(z, f(z))$ contains just one term of maximal total degree in $f(z)$ and its shifts.

It follows from Lemma 2.7, Lemma 2.8, (5.1) and (5.3) that

$$m(r, \tilde{a}_k(z)) = S(r, f) \quad \text{and} \quad N(r, \tilde{a}_k(z)) = S(r, f) \quad (k = 0, 1, \ldots, p).$$

Therefore,

$$T(r, \tilde{a}_k(z)) = S(r, f) \quad (k = 0, 1, \ldots, p). \quad (5.4)$$

Set

$$\Lambda_s := \left\{ \mu \in J : \sum_{j=1}^{\tau_\mu} m_{\mu,j} = s \right\}, \quad s = 0, 1, \ldots, q.$$ 

Similarly, we also rewrite $K(z, f(z))$ as the form

$$K(z, f(z)) = \bar{b}_q(z) f(z)^q + \bar{b}_{q-1}(z) f(z)^{q-1} + \cdots + \bar{b}_1(z) f(z) + \bar{b}_0(z), \quad (5.5)$$

where

$$\bar{b}_0(z) = \sum_{\mu \in \Delta_0} b_\mu(z) \quad \text{and} \quad \bar{b}_s(z) = \sum_{\mu \in \Lambda_s} b_\mu(z) \prod_{j=1}^{\tau_\mu} \left( \frac{f(z + \beta_{\mu,j})}{f(z)} \right)^{m_{\mu,j}}, \quad s = 1, \ldots, q,$$

and

$$\bar{b}_q(z) = \prod_{j=1}^{\tau_q} \left( \frac{f(z + \beta_{\mu,j})}{f(z)} \right)^{m_{\mu,j}} \neq 0 \quad \text{and} \quad \sum_{j=1}^{\tau_q} m_{\mu,j} = q,$$

and

$$T(r, \bar{b}_s(z)) = S(r, f) \quad (s = 0, 1, \ldots, q). \quad (5.6)$$

It follows from (5.2) and (5.5) that

$$R(z, f(z)) = \frac{S(z, f(z))}{K(z, f(z))} = \frac{\tilde{a}_p(z) f(z)^p + \tilde{a}_{p-1}(z) f(z)^{p-1} + \cdots + \tilde{a}_1(z) f(z) + \tilde{a}_0(z)}{\bar{b}_q(z) f(z)^q + \bar{b}_{q-1}(z) f(z)^{q-1} + \cdots + \bar{b}_1(z) f(z) + \bar{b}_0(z)}.$$ 

Thus, we deduce from Lemma 2.6, (5.4) and (5.6) that

$$T(r, R(z, f(z))) = \deg \{ p, q \} T(r, f) + S(r, f).$$

The proof of Theorem 5.1 is completed. \qed
References