A Dynkin Game Under Knightian Uncertainty *

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Abstract

We study a zero-sum Dynkin game under Knightian uncertainty. The associated Hamilton-Jacobi-Bellman-Isaacs equation takes the form of a semi-linear backward stochastic partial differential variational inequality (SBSPDVI). We establish existence and uniqueness of a strong solution by using the Banach fixed point theorem and a comparison theorem. A solution to the SBSPDVI is used to construct a saddle point of the Dynkin game. In order to establish this verification we use the generalized Itō-Kunita-Wentzell formula developed by Yang and Tang (2011).

1 Introduction

A classical Dynking game is a zero-sum game in which player 1 minimizes the following functional over stopping time $\tau_1$ and player 2 maximizes it over stopping time $\tau_2$

$$E[R(\tau_1, \tau_2)] = E\left[ \int_t^{\tau_1 \wedge \tau_2} f(s) \, ds + \xi_{\tau_1} \, \chi_{\{\tau_1 < \tau_2 \wedge T\}} + \xi_{\tau_2} \, \chi_{\{\tau_2 \leq \tau_1, \tau_2 < T\}} + \xi \, \chi_{\{\tau_1 \wedge \tau_2 = T\}} \right], \quad (1.1)$$

with $\xi \leq \xi$, where $f$ is the running cost (or utility), $\xi$ $(\xi)$, resp.) is the lower (upper, resp.) obstacle for the Dynkin game, $\xi$ is the terminal value, and $E$ is the expectation operator with respect to a given objective probability measure. For the case $f(s) = F(s, X_s)$ and $\xi_s = V(s, X_s)$, $\xi_s = \bar{V}(s, X_s)$ with $X_s$ governed by a stochastic differential equation, a saddle point in closed form has been constructed by Yang and Tang [27].

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In the classical Dynkin game the objective functional is an expectation taken with respect to an objective probability measure. In the real world, however, people usually face situations where it is impossible to assess unique probability measures for the future events. For instance, we do not know the exact probability of a stock index to take a value in a specific interval. Knight [21] called such a situation where one cannot assess a unique probability measure for future events uncertainty in contrast to risk where the future outcome cannot be predicted but one can assess a unique probability for each possible event. Ellsberg [13] provided an experimental evidence that people typically exhibit aversion toward Knightian uncertainty. Gilboa and Schmeidler [15] gave an axiomatic model of aversion to Knightian uncertainty (or ambiguity aversion) where a set of probability measures, instead of a unique probability measure, exists and utility is expressed as a minimum of expectations taken with respect to the probability measures, i.e.,

\[ u = \min_{Q \in \Phi} E^Q v(\cdot), \]

where \( \Phi \) is the set of probability measures and \( v(\cdot) \) is the cardinal utility function. The model is called a multiple priors model. Chen and Epstein [3] have extended the model to continuous time and shown that utility can be expressed as a solution to a backward stochastic differential equation (BSDE). Maccheroni, Marinacci, and Rustichini [22] have extended the multiple priors model to include the robust control framework proposed by Hansen and Sargent [17]. Their preference, which is called a variational preference, admits the following representation:

\[ u = \min_{Q \in \Phi} \{ E^Q v(\cdot) + c(Q) \}, \]

where \( c \) is a function defined on the set \( \Phi \) of probability measures which they call the index of ambiguity aversion. In Chen and Epstein [3] the set \( \Phi \) of probability measures is usually defined as that of probability measures \( Q \) whose densities with respect to \( P \) are given by

\[ \frac{dQ}{dP} := \exp \left\{ -\frac{1}{2} \int_0^T \theta_s^2 ds + \int_0^T \theta_s dW_s \right\} \]

with an adapted process \( \theta \) being square-integrable on \([0, T]\) almost surely and taking values in a given set \( K \). If, as assumed in Cheng and Riedel [5], the index \( c \) takes the following form

\[ c(Q) := \int_0^T g(s, \theta_s) ds \]

for some random function \( g \), then a dynamic variational preference is, in general, given by a solution to the following BSDE (as in Tang and Koo [25]):

\[ Y_t = \int_t^T \left[ f(s, Y_s, Z_s) + \eta(s, Z_s) \right] ds - \int_t^T Z_s dW_s \]

(1.2)

where

\[ \eta(s, z) := \inf_{\theta \in K} [g(s, \theta) - \langle z, \theta \rangle]. \]
In this paper we study a Dynkin game with the functional being specified by a general nonlinear BSDE, which incorporates Knightian uncertainty and the dynamic variational preference (1.2). Such a kind of Dynkin game has been addressed by Cvitanic and Karatzas [7], but their saddle point was of an open form. In the following, we are interested in constructing the saddle point of a closed form, in spirits of Yang and Tang [27].

Our approach appears to be straightforward and intuitive. Time consistency of the assumed preference enables us to apply the dynamic programming principle. By the principle we derive its associated Hamilton-Jacobi-Bellman-Isaacs equation, which takes the form of a semi-linear backward stochastic partial differential variational inequality (SBSPDVI). We show existence and uniqueness of a strong solution to the SBSPDVI. The non-linearity of the problem stems from two aspects: the non-linear differential operator and the variational inequality, which cause difficulties. We use the Banach fixed point theorem to establish the existence of a strong solution. But we deal with the space of weak solutions rather than that of strong solutions, because it is highly difficult to deduce the estimates in the space of strong solutions due to the non-linearity. Via proving a comparison theorem for SBSPDVI, we show existence of a weak solution to the SBSPDVI by iteration. Then we show that the weak solution is in fact regular enough to be a strong one by means of the relationship between SBSPDVI and a backward stochastic partial differential equation (BSPDE). Uniqueness is a consequence of the comparison theorem for SBSPDVI.

The verification theorem is important to connect Dynkin games and the SBSPDVI. We prove that the strong solution to the SBSPDVI is the value of the Dynkin game and give a pair of optimal stopping times. There are at least two difficulties because the problem is investigated with a strong solution in a Non-Markov and Knightian uncertain framework. One difficulty is about how to use the Itô formula. As we all know, the value function, $V$, is required to be deterministic in order to apply the Itô formula to it. The requirement is loosened to allow a stochastic function in the Itô-Kunita-Wentzell formula, but the function needs be $C^2$ with respect to state variable, $x$. In our problem, the strong solution $V$ is random, and only has weak second-order derivatives. We overcome the difficulty by applying the generalized Itô-Kunita-Wentzell formula developed by [27]. The other difficulty arises because the payoff can’t be described as a linear conditional expectation due to Knightian uncertain, so that the problem has a strong non-linear property. Fortunately, the payoff can be represented as a solution to the BSDE in (1.2). So, we can use the properties of BSDEs to overcome the difficulty.

Moreover, we show explicitly that the optimal stopping problem by one agent is an extreme case of a Dynkin game. Optimal stopping problems emerge naturally in many fields of economics, finance and statistics. Individual career changes and voluntary retirements, firms’ capital investments, firms’ entry and exist decisions, exercise of American options or conversion of bonds into shares, sample selections involve optimal stopping [6, 9, 14, 24]. An optimal stopping problem can be considered a Dynkin game where a fictitious second player exists who proposes a value which is beyond the reach of the agent. So, our result generalizes those of previous studies on optimal stopping with ambiguity aversion ([5, 24]).

The paper proceeds as follows. In section 2 we explain our assumptions and notation. In section 3 we provide the verification theorem. In section 4 we prove existence and
uniqueness of a strong solution to the SBSPDVI. In section 5 we show that an optimal stopping problem is an extreme case of a Dynkin game. In section 6 we provide two examples, one concerning a convertible bond and the other concerning optimal retirement.

2 Problem, notation and some assumptions.

In our problem, we suppose that the real probability property of the market can be described as $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, on which two independent standard Brownian motions: $d_1$-dimension $W \triangleq \{W_t\}_{t \geq 0}$ and $d_2$-dimension $B \triangleq \{B_t\}_{t \geq 0}$ are defined, which drive the risk factor.

Denote by $\mathbb{F}^W \triangleq \{\mathcal{F}^W_t\}_{t \geq 0}$ and $\mathbb{F}^B \triangleq \{\mathcal{F}^B_t\}_{t \geq 0}$ the natural filtrations generated by $W$ and $B$, respectively. Assume that they contain all $\mathbb{P}$-null sets in $\mathcal{F}$. Define $\mathbb{F} \triangleq \mathbb{F}^W \vee \mathbb{F}^B$. Denote by $\mathcal{P}$ and $\mathcal{P}^B$ the $\sigma$-algebras of predictable sets in $\Omega \times [0, T]$ associated with $\mathbb{F}$ and $\mathbb{F}^B$, respectively. Denote by $\mathcal{B}(D)$ the Borel $\sigma$-algebra of the domain $D$ in $\mathbb{R}^n$.

Suppose that the state process (for example, the price of the risk asset) $X = (X_1, \ldots, X_n)$ is governed by the following stochastic differential equation (SDE):

$$X_{i,s} = x_i + \int_t^s \beta_i(u, X_u) \, du + \int_t^s \gamma_i(u, X_u) \, dW_u^l + \int_t^s \theta_i(u, X_u) \, dB_u^k,$$

where $i = 1, \ldots, n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Note that here and in the following we use repeated indices for summation. For example, the repeated subscript $l$ implies summation over $l = 1, \ldots, d_1$ and the repeated subscript $k$ implies summation over $k = 1, \ldots, d_2$. And the coefficients $\beta, \gamma,$ and $\theta$ in SDE (2.3) satisfy Assumptions (D1) and (D2) (see below in this section).

Introducing the ambiguity aversive (or aversive to Knightian uncertainty) utility representation by the BSDE in (1.2) into the objective functional (1.1), we get the following BSDE for the payoff:

$$R_t(x; \tau_1, \tau_2) = \int_t^{\tau_1 \wedge \tau_2} \mathcal{G}(u, X_{u}^{t,x}, R_{u}^{t,x}, Y_{u}^{t,x}) \, du - \int_t^{\tau_1 \wedge \tau_2} Y_{u}^{1,l} \, dW_u^l - \int_t^{\tau_1 \wedge \tau_2} Y_{u}^{2,k} \, dB_u^k + \mathcal{V}_{\tau_1}(X_{\tau_1}^{t,x}) \chi_{\tau_1 < \tau_2 \wedge T} + \mathcal{V}_{\tau_2}(X_{\tau_2}^{t,x}) \chi_{\tau_2 \leq \tau_1 \wedge \tau_2 < T} + \phi(X_{\tau_2}^{t,x}) \chi_{\tau_1 \wedge \tau_2 = T}. \quad (2.4)$$

Here $\mathcal{U}_{t,T}$ is the class of all $\mathbb{F}$-stopping times which take values in $[t, T]$ and $(\tau_1, \tau_2) \in \mathcal{U}_{t,T} \times \mathcal{U}_{t,T}$, which consists of the strategies of both agents. The above BSDE has a non-linear generator $\mathcal{G}$ with the terminal value being the sum of the last three terms on the right-hand side of the equation. The solution $(R, Y^1, Y^2)$ obviously depends on the pair of stopping times $(\tau_1, \tau_2)$, and its first component $R^{t,x}_l$ will be denoted by $R_t(x; \tau_1, \tau_2)$. Consider the following Dynkin game under Knightian uncertainty: each player tries either to minimize or to maximize $R_t(x; \tau_1, \tau_2)$.

Throughout this paper, we assume that the running payoff $G$ is a $\mathcal{P}^B \times \mathcal{B}(\mathbb{R}^{n+1+(d_1+d_2)})$-measurable random field taking values in $\mathbb{R}$, and satisfies Assumption (D3)$_{\lambda}$ (see below in this section). The terminal payoffs $\mathcal{V}$ and $\mathcal{V}$ are $\mathcal{P}^B \times \mathcal{B}(\mathbb{R}^n)$-measurable random fields taking values in $\mathbb{R}$, and $\phi$ is a $\mathcal{F}_{t,T}^B \times \mathcal{B}(\mathbb{R}^n)$-measurable random field taking values in $\mathbb{R}$. They satisfy some proper assumptions.
In this paper, we consider the following Nash equilibrium of the non-Markovian zero-sum Dynkin game (denoted by $\mathcal{D}_{tx}$ hereafter):

Find a pair $(\tau_1^*, \tau_2^*) \in \mathcal{U}_{t,T} \times \mathcal{U}_{t,T}$ such that the following inequalities hold

$$R_t(x; \tau_1^*, \tau_2^*) \geq R_t(x; \tau_1^*, \tau_2^*) \geq R_t(x; \tau_1, \tau_2^*) \text{ for any } \tau_1, \tau_2 \in \mathcal{U}_{t,T}.$$

Such a pair $(\tau_1^*, \tau_2^*)$, if it exists, is called a Nash equilibrium point or saddle point of Problem $\mathcal{D}_{tx}$, and the random variable $V_t(x) \triangleq R_t(x; \tau_1^*, \tau_2^*)$ is called the value of Problem $\mathcal{D}_{tx}$. In this case, we have

$$V_t(x) = \text{ess.inf} \sup_{\tau_2 \in \mathcal{U}_{t,T}} R_t(x; \tau_1, \tau_2) = \sup_{\tau_1 \in \mathcal{U}_{t,T}} \text{ess.inf} R_t(x; \tau_1, \tau_2).$$

The value $V_t(x)$ is unique if it exists. In general, a Nash equilibrium point may not be unique, and here it always means the smallest one.

In order to facilitate the discussion, we introduce the following notation:

Denote by $\mathbb{N}$ and $\mathbb{N}_+$ the set of all nonnegative and positive integers, respectively. Denote by $E$ a Euclidean space like $\mathbb{R}$ or $\mathbb{R}^n$, $\mathbb{R}^{n \times d_1}$, $\mathbb{R}^{n \times d_2}$, and $\mathbb{R}^{n \times n}$. Moreover, for any $x \in \mathbb{R}^n$, $\gamma \in \mathbb{R}^{n \times d_1}$, $\theta \in \mathbb{R}^{n \times d_2}$ and $a \in \mathbb{R}^{n \times n}$, define

$$|x| \triangleq \left( \sum_{i=1}^{n} x_i^2 \right)^\frac{1}{2}, \quad |\gamma| \triangleq \left( \sum_{i=1}^{n} \sum_{l=1}^{d_1} \gamma_{il}^2 \right)^\frac{1}{2},$$

$$|\theta| \triangleq \left( \sum_{i=1}^{n} \sum_{k=1}^{d_2} \theta_{ik}^2 \right)^\frac{1}{2}, \quad \text{and } |a| \triangleq \left( \sum_{i,j=1}^{n} a_{ij}^2 \right)^\frac{1}{2}.$$  

Define $D_i \triangleq \partial x_i$; $D_{ij} \triangleq \partial x_i x_j$. Denote by $D\eta$ the gradient of the function $\eta : E \rightarrow \mathbb{R}$.

For an integer $k \in \mathbb{N}$, $p \in [1, +\infty)$, $q \in [1, +\infty)$, $\lambda \in [0, +\infty)$, and a positive number $T$, we introduce the following spaces:

- $C^k$, the set of all functions $\eta : \mathbb{R}^n \rightarrow E$ such that $\eta$ and $D^{|\alpha|} \eta$ are continuous for all $1 \leq |\alpha| \leq k$;
- $C_0^k$, the set of all functions in $C^k$ with compact support in $\mathbb{R}^n$;
- $H_{\lambda}^{k,p}$, the completion of $C^k$ under the norm

$$|\eta|_{k,p;\lambda} \triangleq \left( \int_{\mathbb{R}^n} |\eta|^p e^{-\lambda|x|} \, dx + \sum_{|\alpha|=1}^{k} \int_{\mathbb{R}^n} |D^\alpha \eta|^p e^{-\lambda|x|} \, dx \right)^\frac{1}{p};$$

- $L_{\lambda}^{k,p}$, the set of all $H_{\lambda}^{k,p}$-valued and $\mathbb{F}_T$-measurable random variables such that $\mathbb{E}(|\varphi|^p_{k,p;\lambda}) < \infty$;
- $\mathcal{L}^p$, the set of all $\mathcal{P}$-predictable stochastic processes taking values in $E$ with the norm

$$\|X\|_p \triangleq \left[ \mathbb{E} \left( \int_0^T |X_t|^p \, dt \right) \right]^\frac{1}{p};$$
\* \( S^p \), the set of all path continuous processes in \( \mathcal{L}^p \) with the norm
\[
\|X\|_p \triangleq \left[ \mathbb{E} \left( \sup_{t \in [0,T]} |X_t|^p \right) \right]^{\frac{1}{p}};
\]
\* \( \mathbb{H}^{k,p}_\lambda \), the set of all \( \mathcal{P}^B \)-predictable stochastic processes with values in \( H^{k,p}_\lambda \) with the norm
\[
\|V\|_{k,p;\lambda} \triangleq \left[ \mathbb{E} \left( \int_0^T |V_t|^p_{k,p;\lambda} \, dt \right) \right]^{\frac{1}{p}};
\]
\* \( S^{k,p}_\lambda \), the set of all path continuous stochastic processes in \( \mathbb{H}^{k,p}_\lambda \) equipped with the norm
\[
\|V\|_{k,p;\lambda} \triangleq \left[ \mathbb{E} \left( \sup_{t \in [0,T]} |V_t|^p_{k,p;\lambda} \right) \right]^{\frac{1}{p}};
\]

The space notation \( H^{k,p}_0, \mathbb{L}^{k,p}_0, \mathbb{H}^{k,p}_0, S^{k,p}_0, |\eta|_{k,p;0}, \|V\|_{k,p;0}, \) and \( \|V\|_{k,p} \) will be abbreviated as \( H^{k,p}, \mathbb{L}^{k,p}, \mathbb{H}^{k,p}, S^{k,p}, |\eta|_{k,p}, \|V\|_{k,p} \) and \( \|V\|_{k,p} \) if there is no confusion.

Throughout this paper, we assume that the coefficients \( \beta, \gamma, \) and \( \theta \) satisfy Assumptions (D1) and (D2):

**Assumption (D1) (Boundedness).** \( \beta, \gamma, \theta \) are \( \mathcal{P}^B \times \mathcal{B}(\mathbb{R}^n) \)-measurable with values in \( \mathbb{R}^n, \mathbb{R}^{n \times d_1}, \mathbb{R}^{n \times d_2} \), respectively. Moreover, they are bounded by a positive constant \( K \), i.e.,
\[
|\beta(\cdot, x)| + |\gamma(\cdot, x)| + |\theta(\cdot, x)| \leq K \text{ for any } x \in \mathbb{R}^n, \text{ a.e. in } \Omega \times (0,T).
\]

**Assumption (D2) (Lipschitz continuity and nondegeneracy).** \( \beta, \gamma, \) and \( \theta \) are uniformly Lipschitz in \( x \), i.e., there exists a positive constant \( K \) such that
\[
|\beta(\cdot, x_1) - \beta(\cdot, x_2)| + |\gamma(\cdot, x_1) - \gamma(\cdot, x_2)| + |\theta(\cdot, x_1) - \theta(\cdot, x_2)| \leq K |x_1 - x_2|
\]
for any \( x_1, x_2 \in \mathbb{R}^n \) almost everywhere in \( \Omega \times (0,T) \). Moreover, there exists a positive constant \( \kappa \) such that
\[
\gamma_{ij} \gamma_{kl} \xi_i \xi_j \geq \kappa |\xi|^2 \quad \forall \xi \in \mathbb{R}^n \text{ a.e. in } \Omega \times (0,T) \times \mathbb{R}^n.
\]

It is clear that SDE (2.3) has a unique strong solution \( X \in S^p \) for any \( p \geq 1 \).

Moreover, we assume the running payoff \( G \) satisfies:

**Assumption (D3) (Regularity).** \( G \) is uniformly Lipschitz in \( (r,y) \in \mathbb{R}^{1+(d_1+d_2)} \), i.e., there exists a positive constant \( K \) such that
\[
|G(\cdot, r_1, y_1) - G(\cdot, r_2, y_2)| \leq K (|r_1 - r_2| + |y_1 - y_2|)
\]
for any \( (r_1, y_1), (r_2, y_2) \in \mathbb{R}^{1+(d_1+d_2)} \) almost everywhere in \( \Omega \times (0,T) \times \mathbb{R}^n \).

Moreover, \( G(\cdot, 0, 0, 0) \in \mathcal{L}^2, G(\cdot, 0, 0) \in \mathbb{H}^{k,p}_\lambda \). And \( G \) grows in a polynomial way with respect to \( x \), i.e., there exist a positive constant \( K \) and a number \( m \in \mathbb{N}_+ \) such that
\[
|G(\cdot, x, \cdot) - G(\cdot, 0, \cdot)| \leq K (1 + |x|^m) \quad \forall x \in \mathbb{R}^n \text{ a.e. in } \Omega \times (0,T) \times \mathbb{R}^{1+d_1+d_2}.
\]

The assumption notation (D3) will be abbreviated as (D3) if there is no confusion.

**Remark 2.1.** Since \( X \in S^p \) for any \( p \geq 1 \), then \( G(\cdot, X, 0, 0) \in \mathcal{L}^2 \). Hence, BSDE (2.4) has a unique solution \( (R^{t,x;\tau_1,\tau_2}, Y^{t,x;\tau_1,\tau_2}) \) for any fixed \( (t, x; \tau_1, \tau_2) \) under some proper assumptions imposed on \( \underline{V}, \overline{V}, \varphi \).
3 Verification theorem.

In this section, we prove the verification theorem that the Nash equilibrium point and the value of the Dynkin game are characterized by the strong solution of SBSPDVI:

\[
\begin{cases}
    dV_t = -(LV_t + M^k Z^k_t + f_t + F(t, V_t, DV_t, Z_t)) \, dt + Z^k_t \, dB^k_t & \text{if } V_t < V_t < \nabla_t; \\
    dV_t \leq -(LV_t + M^k Z^k_t + f_t + F(t, V_t, DV_t, Z_t)) \, dt + Z^k_t \, dB^k_t & \text{if } V_t = V_t; \\
    dV_t \geq -(LV_t + M^k Z^k_t + f_t + F(t, V_t, DV_t, Z_t)) \, dt + Z^k_t \, dB^k_t & \text{if } V_t = \nabla_t; \\
    V_T = \varphi,
\end{cases}
\]

where the repeated superscript \(k\) is summed from 1 to \(d_2\), and

\[
\mathcal{L} \triangleq a^{ij} D_{ij} + b^i D_i + c, \quad M^k \triangleq \sigma^{ik} D_i + \mu^k, \quad i, j = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, d_2. \quad (3.2)
\]

**Remark 3.1.** In fact, the free term \(f\) can be included in \(F\). But we keep it because it is convenient to give some estimates for the SBSPDVI.

In this paper, we maybe consider the following assumptions:

**Assumption (V1)** (Boundedness). Functions \(a, b, c, \sigma, \mu\) are \(P^B \times \mathcal{B}(\mathbb{R}^n)\)-measurable, and are bounded by a constant \(K\), taking values in the set of real symmetric matrices, in the spaces \(\mathbb{R}^n, \mathbb{R}, \mathbb{R}^{n \times d_2}, \mathbb{R}^{d_2}\), respectively. \(Da\) and \(D\sigma\) exist almost everywhere and are bounded by \(K\).

**Assumption (V2)** (Superparabolicity). There exist two positive constants \(\kappa\) and \(K\) such that:

\[
\kappa |\xi|^2 + |\sigma^* \xi|^2 \leq 2 \xi^*(a^{ij})\xi \leq K |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n, \text{ a.e. in } \Omega \times (0, T) \times \mathbb{R}^n.
\]

**Assumption (V3)** (Regularity). The non-linear function \(F\) is uniformly Lipschitz in \((v, y, z) \in \mathbb{R}^{1+n+d_2}\), i.e., there exists a positive constant \(K\) such that

\[
|F(\cdot, v_1, y_1, z_1) - F(\cdot, v_2, y_2, z_2)| \leq K (|v_1 - v_2| + |y_1 - y_2| + |z_1 - z_2|)
\]

for any \((v_1, y_1, z_1), (v_2, y_2, z_2) \in \mathbb{R}^{1+n+d_2}\), almost everywhere in \(\Omega \times (0, T) \times \mathbb{R}^n\). Moreover, \(F(\cdot, 0, 0, 0) \in H^{0,2}_{\lambda}\).

**Assumption (V4)** (Regularity). \(f \in H^{0,2}_{\lambda}, \varphi \in L^{1,2}_{\lambda}, V, \nabla \in H^{1,2}_{\lambda}\). And \(V\) and \(\nabla\) are continuous semimartingales of the following form

\[
dV_t = -g_t \, dt + Z^k_t \, dB^k_t, \quad d\nabla_t = -\overline{g}_t \, dt + \overline{Z}^k_t \, dB^k_t
\]

for some \((g, \overline{g}, Z, \overline{Z}) \in H^{0,2}_{\lambda} \times H^{0,2}_{\lambda} \times H^{0,2}_{\lambda} \times H^{0,2}_{\lambda}\). Moreover, there exists a nonnegative random field \(h \in H^{0,2}_{\lambda}\) such that

\[
\mathcal{L} V + M^k Z^k - g + h \geq 0 \quad \text{and} \quad \mathcal{L} \nabla + M^k \overline{Z}^k - \overline{g} - h \leq 0
\]

hold in the sense of distribution. That is, for any nonnegative function \(\eta \in C_0^2(\mathbb{R}^n)\), we have

\[
T(V, Z, g, h, \eta) \geq 0 \quad \text{and} \quad T(\nabla, \overline{Z}, \overline{g}, -h, \eta) \leq 0 \quad \text{a.e. in } \Omega \times [0, T],
\]

where \(T(X, Y, Z, W, \eta)\) is defined by

\[
T(X, Y, Z, W, \eta) = \int_{\Omega \times [0, T]} \eta (X, Y, Z, W) \, dV_t + \int_{\Omega \times [0, T]} \eta (X, Y, Z, W) \, d\nabla_t - \int_{\Omega \times [0, T]} \eta (X, Y, Z, W) \, dB_t^k - \int_{\Omega \times [0, T]} \eta (X, Y, Z, W) \, dB_t^k
\]
where
\[
T(V, Z, g, h, \eta) \triangleq -\int_{\mathbb{R}^n} \left( a^{ij} D_i V + \sigma^{jk} Z^k \right) D_j \eta \, dx + \int_{\mathbb{R}^n} (h - g) \eta \, dx \\
+ \int_{\mathbb{R}^n} \left[ (b^i - D_j a^{ij}) D_i V + c V + \left( \mu^k - D_i \sigma^{ik} \right) Z^k \right] \eta \, dx.
\]

**Assumption (V5) (Compatibility).** \( V \leq \bar{V}, \ V_T \leq \varphi \leq \bar{V}_T \) and
\[
\left( \mathcal{L} V_t + \mathcal{M}^k Z_t^k + f_t + F(t, \cdot, V_t, D V_t, Z_t) - g_t \right) \chi_{\{V = \bar{V}\}} = 0 \text{ a.e. in } \Omega \times \overline{Q},
\]
where \( Q \triangleq [0, T) \times \mathbb{R}^n \).

**Assumption (V4)\lambda.** \( f \in \mathbb{H}_\lambda^{2, 2}, \varphi \in \mathbb{L}_\lambda^{1, 2}, \) and \( V \) and \( \bar{V} \) have the following representation:
\[
dV_t = -g_t \, dt + Z_t^k dB_t^k, \quad d\bar{V}_t = -\bar{g}_t \, dt + \bar{Z}_t^k d\bar{B}_t^k
\]
with \( V, \bar{V} \in \mathbb{H}_\lambda^{2, 2}, Z, \bar{Z} \in \mathbb{H}_\lambda^{1, 2}, \) and \( g, \bar{g} \in \mathbb{H}_\lambda^{0, 2} \).

Assumptions (V3)\( _0, \) (V4)\( _0 \) and (V4)'\( _0 \) will be abbreviated as (V3), (V4) and (V4)' if there is no confusion.

**Remark 3.2.** According to [27, Proposition 4.1], Assumption (V4)' implies Assumption (V4). Therefore, \( V, \bar{V} \) and \( \varphi \) satisfy Assumption (V4) if they satisfy Assumption (V4)'. And Assumption (V4)'\( _\lambda \) implies Assumption (V4)'\( _\lambda \) since \( \eta \in C^0_0(\mathbb{R}^n) \).

The strong solution of SBSPDVI (3.1) is defined as follows:

**Definition 3.1.** If the four-tuple \( (V, Z, k^+, k^-) \in \mathbb{H}_\lambda^{2, 2} \times \mathbb{H}_\lambda^{1, 2} \times \mathbb{H}_\lambda^{0, 2} \times \mathbb{H}_\lambda^{0, 2} \) with some nonnegative number \( \lambda \), and satisfies:
\[
\begin{align*}
V_t = & \ \varphi + \int_t^T (\mathcal{L} s + \mathcal{M}^k Z^k_s + f_s + F(s, \cdot, V_s, D V_s, Z_s) + k^+_s - k^-_s) \, ds - \int_t^T Z^k_s \, dB^k_s \\
& \text{a.e. } x \in \mathbb{R}^n \text{ for all } t \in [0, T] \text{ and a.s. in } \Omega; \\
V & \leq V \leq \bar{V}, \quad k^\pm \geq 0 \text{ a.e. in } \Omega \times Q; \\
\int_0^T (V_t - V_s) \, k^+_s \, dt = & \int_0^T (\bar{V}_t - \bar{V}_s) \, k^-_s \, dt = 0 \text{ a.e. in } \Omega \times \mathbb{R}^n.
\end{align*}
\]

Then \( (V, Z, k^+, k^-) \) is called a strong solution of BSPDVI (3.1).

Recalling the generalized Itô formula in [27], we have

**Lemma 3.1.** Suppose that the random function \( V : \Omega \times [0, T] \times \mathbb{R}^n \to \mathbb{R} \) satisfies the following: \( V(x) \) is a continuous semimartingale of form:
\[
V_t(x) = V_0(x) + \int_0^t U_s(x) \, ds + \int_0^t Z^k_s(x) \, dB^k_s \quad \text{a.e. } x \in \mathbb{R}^n
\]
for every $t \in [0, T]$ and almost surely $\omega \in \Omega$, such that $V \in \mathbb{H}^{2,2}$, $Z \in \mathbb{H}^{1,2}$, and $U \in \mathbb{H}^{0,2}$. Let $X$ be a continuous semi-martingale of form (2.3), and Assumptions (D1) and (D2) be satisfied. Then we have

$$V_t(X_t) = V_0(X_0) + \int_0^t (LV_s + M^kZ_s^k + U_s)(X_s) \, ds + \int_0^t (Z_s^k + M^kV_s)(X_s) \, dB_s^k$$

$$+ \int_0^t (N^lV_s)(X_s) \, dW_s^l,$$

where the repeated superscript $l$ is summed from $1$ to $d_1$ and the repeated superscript $k$ is summed from $1$ to $d_2$, and

$$L \triangleq \frac{1}{2} (\gamma_{ij} + \theta_{ik}\theta_{jk}) D_{ij} + \beta_i D_i, \quad M^k \triangleq \theta_{ik} D_i, \quad N^l \triangleq \gamma_{il} D_i$$

(3.6) for any $l = 1, 2, \ldots, d_2, k = 1, 2, \ldots, d_1$, and the repeated superscript $i, j$ is summed from $1$ to $n$.

We have the following verification theorem for Problem $\mathcal{D}_{tx}$.

**Theorem 3.2. (Verification)** Let Assumptions (D1)-(D3), (V4) and (V5) be satisfied and $(t,x) \in Q$. Let $X$ be the solution of SDE (2.3). And assume that $V(X) \triangleq \mathcal{V}(X)$, $\mathcal{V}(X) \triangleq \mathcal{V}(\cdot, X) \in S^2$ and $\varphi(X_T) \in L^2$. Moreover, let $(V,Z,k^+,k^-)$ be the strong solution of SBSPDVI (3.1) with

$$f \equiv 0, \quad F(\cdot, v, y, z) = G(\cdot, v, \gamma^{1k}y_k, \ldots, \gamma^{d_1k}y_k, z_1 + \sigma^{1k}y_k, \ldots, z_{d_2} + \sigma^{d_2k}y_k)$$

(3.7) and

$$a^{ij} \triangleq \frac{1}{2} \Big( \sum_{l=1}^{d_1} \gamma_{il}\gamma_{jl} + \sum_{l=1}^{d_2} \theta_{il}\theta_{jl} \Big), \quad b^i \triangleq \beta_i, \quad c \triangleq 0, \quad \sigma^{ik} \triangleq \theta_{ik}, \quad \mu_k \triangleq 0$$

(3.8) for any $i, j = 1, \ldots, n$ and $k = 1, \ldots, d_2$. Then $V(t,x)$ is the value of Problem $\mathcal{D}_{tx}$. Define

$$\tau_1^* \triangleq \inf \{ s \in [t, T] : V_s(X_s) = \mathcal{V}_s(X_s) \} \wedge T$$

and

$$\tau_2^* \triangleq \inf \{ s \in [t, T] : V_s(X_s) = \mathcal{V}_s(X_s) \} \wedge T.$$

Then, $(\tau_1^*, \tau_2^*)$ is a Nash equilibrium point of Problem $\mathcal{D}_{tx}$.

**Proof of Theorem 3.2.** It is sufficient to prove the following for any $\tau_1, \tau_2 \in \mathcal{U}_{t,T}$,

$$R_t(x; \tau_1^*, \tau_2) \geq V_t(x) \geq R_t(x; \tau_1, \tau_2^*) \quad \text{a.s. in } \Omega,$$

with the equality holding true in the first inequality if $\tau_2 = \tau_2^*$ and in the second inequality if $\tau_1 = \tau_1^*$. In what follows, we only prove the second inequality and the first one can be proved in a symmetric way.
According to [27, Theorem 3.1], we deduce that \( V(X) - V(X) \) and \( \nabla(X) - V(X) \) are stochastic processes with continuous paths. So, we have that

\[
\chi_{\{\tau^*_1 < T\}} V_{\tau^*_1}^1(X_{\tau^*_1}) = \chi_{\{\tau^*_1 < T\}} \nabla_{\tau^*_1}^1(X_{\tau^*_1}) \quad \text{and} \quad \chi_{\{\tau^*_2 < T\}} V_{\tau^*_2}^2(X_{\tau^*_2}) = \chi_{\{\tau^*_2 < T\}} \nabla_{\tau^*_2}^2(X_{\tau^*_2})
\]

almost everywhere in \( \Omega \times Q \), and

\[
V_s(X_s) - V_s(X_s) > 0 \quad \text{for any} \quad t \leq s < \tau^*_1 \quad \text{and} \quad \nabla_s(X_s) - V_s(X_s) > 0 \quad \text{for any} \quad t \leq s < \tau^*_2
\]

almost surely in \( \Omega \). Moreover, according to the method in [27], we can deduce the following equalities from the equalities on the sixth line in Definition 3.1,

\[
k^+_s(X_s) \chi_{\{s < \tau^*_1\}} = 0, \quad k^-_s(X_s) \chi_{\{s < \tau^*_2\}} = 0 \quad \text{a.e. in} \quad \Omega \times (0, T),
\]

and for any \( \tau_1, \tau_2 \in \mathcal{U}_{t,T} \) satisfying \( \tau_1 \leq \tau^*_1, \tau_2 \leq \tau^*_2 \), the following hold

\[
\int_t^{\tau_1} k^+_s(X_s) \, ds = 0, \quad \text{and} \quad \int_t^{\tau_2} k^-_s(X_s) \, ds = 0, \quad \text{a.s. in} \quad \Omega.
\]

(3.9)

Define

\[
\mathcal{R}_t(x; \tau_1, \tau_2) \triangleq \nabla_{\tau_1}^1(X_{\tau_1}^t, x) \chi_{\{\tau_1 < \tau_2 \wedge T\}} + \nabla_{\tau_2}^2(X_{\tau_2}^t, x) \chi_{\{\tau_2 \leq \tau_1, \tau_2 < T\}} + \varphi(X_T^t, x) \chi_{\{\tau_2 = T\}}.
\]

On the event \( \{\tau_1 \in \mathcal{U}_{t,T} : \tau_1 \geq \tau^*_2\} \), applying Lemma 3.1, we have

\[
\mathcal{R}_t(x; \tau_1, \tau^*_2) = \nabla_{\tau_2}^2(X_{\tau_2}^t, x) \chi_{\{\tau_2 < T\}} + \varphi(X_T^t, x) \chi_{\{\tau_2 = T\}} = V_{\tau_2}^T(X_{\tau_2}^t, x)
\]

\[
= V_t(X_t^t) - \int_t^{\tau_2} \left[ \mathcal{L}u + M^kZ^k_u + F(u, x, V_u, D^2V_u, Z_u) + k^+_u - k^-_u \right] (X_u^t) \, du
\]

\[
+ \int_t^{\tau_2} (LV_u + M^kZ^k_u)(X_u^t) \, du + M_1(\tau^*_2)
\]

where we have used (3.7), (3.8) and (3.9), and

\[
M_1(\tau) \triangleq \int_t^\tau N^tV_u(X_u^t) \, dW_u^t + \int_t^\tau (Z_u^k + M^kV_u)(X_u^t) \, dB_u^k.
\]

Recalling \( k^+ \geq 0 \), we have

\[
\mathcal{R}_t(x; \tau_1, \tau^*_2) \leq V_t(x) - \int_t^{\tau_2} G(u, x, V_u, N^tV_u, Z_u + MV_u)(X_u^t) \, du + M_1(\tau^*_2) \quad \text{a.s. in} \quad \Omega,
\]

with the equality holding true if \( \tau_1 = \tau^*_1 \), which follows from \( \tau_2^* \leq \tau^*_1 \) and (3.9).

On the event \( \{\tau \in \mathcal{U}_{t,T} : \tau_1 < \tau^*_2\} \), in a similar way, we have

\[
\mathcal{R}_t(x; \tau_1, \tau^*_2) = \nabla_{\tau_1}^1(X_{\tau_1}^t, x) \leq V_{\tau_1}^1(X_{\tau_1}^t, x)
\]

\[
\leq V_t(x) - \int_t^{\tau_1} G(u, x, V_u, N^tV_u, Z_u + MV_u)(X_u^t) \, du + M_1(\tau_1) \quad \text{a.s. in} \quad \Omega,
\]

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with the equality holding true if $\tau_1 = \tau_1^*$. So, we obtain that
\[
\mathcal{V}_{\tau_1}(X_{\tau_1}^{t,x}) \chi_{\{\tau_1 < \tau_2^* \wedge T\}} + \nabla_{\tau_2^*}(X_{\tau_{2}^*}^{t,x}) \chi_{\{\tau_2^* \leq \tau_1, \tau_2^* < T\}} + \varphi(X_{T}^{t,x}) \chi_{\{\tau_1 \wedge \tau_2^* = T\}}
\]
\[
= \mathcal{R}(x; \tau_1, \tau_2^*) \leq V_t(x) - \int_t^{\tau_1 \wedge \tau_2^*} G(u, x, V_u, N V_u, Z_u + M V_u)(X_u^{t,x}) d u + M_1(\tau_1 \wedge \tau_2^*),
\]
after almost surely with the equality holding true if $\tau_1 = \tau_1^*$. This means that
\[
V_t(x) \geq \int_t^{\tau_1 \wedge \tau_2^*} G(u, x, V_u, N V_u, Z_u + M V_u)(X_u^{t,x}) d u - \int_t^{\tau_1 \wedge \tau_2^*} N V_u(X_u^{t,x}) d W_u
\]
\[
- \int_t^{\tau_1 \wedge \tau_2^*} (Z_u^k + M V_u)(X_u^{t,x}) dB_u + \nabla_{\tau_1}(X_{\tau_1}^{t,x}) \chi_{\{\tau_1 < \tau_2^* \wedge T\}}
\]
\[
+ \nabla_{\tau_2^*}(X_{\tau_2^*}^{t,x}) \chi_{\{\tau_2^* \leq \tau_1, \tau_2^* < T\}} + \varphi(X_{T}^{t,x}) \chi_{\{\tau_1 \wedge \tau_2^* = T\}}
\]
after almost surely with the equality holding true if $\tau_1 = \tau_1^*$.

From the comparison principle for BSDE, we deduce that
\[
R_t(x; \tau_1, \tau_2^*) \leq V_t(x) \text{ for any } \tau_1 \in \mathcal{U}_{t,T} \text{ a.s. in } \Omega,
\]
with the equality holding true if $\tau_1 = \tau_1^*$. The proof is then complete.

\[
\square
\]

4 Strong solution of SBSPDVI (3.1): existence and uniqueness, and comparison theorem.

In the section, we use the Banach fixed point theory and the comparison theory for the SBSPDVI to established the existence and uniqueness of the strong solution of SBSPDVI (3.1).

First, we review an existing estimate for the following semi-linear backward stochastic partial differential equation (SBSPDE, for short) [10, Theorem 5.3]:
\[
\begin{cases}
    dV_t = - (\mathcal{L} V_t + M^k Z_t^k + f_t + F(t, \cdot, V_t, D V_t, Z_t)) \, dt + Z_t^k \, dB_t^k; \\
    V_T = \varphi.
\end{cases}
\tag{4.1}
\]

Lemma 4.1. Let the differential operators $\mathcal{L}$ and $M$ satisfy Assumptions (V1) and (V2), the nonlinear term $F$ satisfy Assumption (V3), the free term $f \in H^{0,2}$, and the terminal value $\varphi \in L^{1,2}$. Then SBSPDE (4.1) has a unique strong solution $(V, Z)$. Moreover, it satisfies the following estimate:
\[
\|V\|_{2,2} + ||V||_{1,2} + \|Z\|_{1,2} \leq C(\kappa, K, T) \left( \mathbb{E} \left[ \|\varphi\|_{1,2} \right] + ||f||_{0,2} + \|F(\cdot, 0, 0, 0)\|_{0,2} \right),
\]
where $C(\kappa, K, T)$ is a constant depending on $\kappa$ and $K, T$.

Then we recall an estimate for linear backward stochastic partial differential equations (BSPDE, for short) [27, Lemma 5.1]:

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Lemma 4.2. Let the differential operators $\mathcal{L}$ and $\mathcal{M}$ satisfy Assumptions (V1) and (V2), and $F \equiv 0$, $f \in \mathbb{H}^{0,2}$, $\varphi \in \mathbb{L}^{1,2}$. Then the strong solution $(V, Z)$ of BSPDE (4.1) satisfies the following estimate:

$$
\|V\|_{1,2}^2 + \|V\|_{0,2}^2 + \|Z\chi_{\{V \leq 0\}}\|_{0,2}^2 \leq C(\kappa, K, T) \mathbb{E} \left( |\varphi^-|_{0,2}^2 + \int_0^T \int_{\mathbb{R}^n} f_s^- V_s^- \, dx \, ds \right),
$$

where $C(\kappa, K, T)$ is a constant increasing with respect to $K, T$, and decreasing with respect to $\kappa$.

We then have the following estimate for SBSPDVI (3.1):

Lemma 4.3. Let Assumption (V1)-(V5) be satisfied, and $(V_i, Z_i, k_i^+, k_i^-) \in \mathbb{H}^{2,2} \times \mathbb{H}^{1,2} \times \mathbb{H}^{0,2} \times \mathbb{H}^{0,2}$ be the strong solution of SBSPDVI (3.1) with $(f, \varphi, V, V) \triangleq (f_i, \varphi_i, V_i, V_i)$ for $i = 1, 2$. Suppose $\nabla V_1 \geq \nabla V_2$ and $\nabla V_1 \geq \nabla V_2$. Denote

$$
\Delta f = f_1 - f_2, \quad \Delta \varphi = \varphi_1 - \varphi_2, \quad \Delta V = V_1 - V_2, \quad \Delta Z = Z_1 - Z_2.
$$

Then we have that

$$
\|(\Delta V)^-\|_{1,2}^2 + \|(\Delta V)^-\|_{0,2}^2 + \|\Delta Z\chi_{\{\Delta V \leq 0\}}\|_{0,2}^2 
\leq C(\kappa, K, T) \mathbb{E} \left( |(\Delta \varphi)^-|_{0,2}^2 + \int_0^T \int_{\mathbb{R}^n} (\Delta f_s)^- (\Delta V_s)^- \, dx \, ds \right),
$$

$$
\|(\Delta V)^-\|_{1,2}^2 + \|(\Delta V)^-\|_{0,2}^2 + \|\Delta Z\|_{0,2}^2 
\leq C(\kappa, K, T) \mathbb{E} \left( |\Delta \varphi^-|_{0,2}^2 + \int_0^T \int_{\mathbb{R}^n} (\Delta f_s \Delta V_s)^+ \, dx \, ds \right),
$$

$$
\|(\Delta V)^-\|_{1,2}^2 + \|(\Delta V)^-\|_{0,2}^2 + \|\Delta Z\|_{0,2}^2 \leq C(\kappa, K, T) \mathbb{E} \left( |\Delta \varphi^-|_{0,2}^2 + \|\Delta f\|_{0,2}^2 \right),
$$

where $C(\kappa, K, T)$ is a constant increasing with respect to $K, T$, and decreasing with respect to $\kappa$.

Proof. Denote $\Delta k^+ \triangleq k_1^+ - k_2^+$, $\Delta k^- \triangleq k_1^- - k_2^-$, $\Delta F_s = F(s, \cdot, V_{1,s}, D V_{1,s}, Z_{1,s}) - F(s, \cdot, V_{2,s}, D V_{2,s}, Z_{2,s})$. Then $\Delta V$ satisfies the following BSPDE:

$$
\Delta V_t = \Delta \varphi + \int_t^T (\mathcal{L} \Delta V_s + \mathcal{M}^k \Delta Z_s^k + \Delta f_s + \Delta F_s + \Delta k^+ - \Delta k^-) \, ds - \int_t^T \Delta Z_s^k \, dB_s^k.
$$

In view of Lemma 4.2 and $k_1^+, k_2^+ \geq 0$, we have that

$$
\|(\Delta V)^-\|_{1,2}^2 + \|(\Delta V)^-\|_{0,2}^2 + \|\Delta Z\chi_{\{\Delta V \leq 0\}}\|_{0,2}^2 
\leq C \mathbb{E} \left( |(\Delta \varphi)^-|_{0,2}^2 + \int_0^T \int_{\mathbb{R}^n} (\Delta f_s + \Delta F_s + \Delta k^+ - \Delta k^-)^- (\Delta V_s)^- \, dx \, ds \right),
$$

$$
\leq C \mathbb{E} \left( |(\Delta \varphi)^-|_{0,2}^2 + \int_0^T \int_{\mathbb{R}^n} \left[ (\Delta f_s)^- + (\Delta F_s)^- + k_{1,s}^- + k_{2,s}^+ \right] (\Delta V_s)^- \, dx \, ds \right).
$$
Hence, we deduce that
\[
\int_0^T \int_{\mathbb{R}^n} (\Delta F_s)^- (\Delta V_s)^- \, dx \, ds \\
\leq C_K \int_0^T \int_{\mathbb{R}^n} \left[ |\Delta V_s| + |D \Delta V_s| + |\Delta Z_s| \right] (\Delta V_s)^- \, dx \, ds \\
\leq \frac{1}{2} \| (\Delta V)^- \|_{1,2}^2 + \frac{1}{2} \| \Delta Z \chi_{\{\Delta V \leq 0\}} \|_{0,2}^2 + C_K \| (\Delta V)^- \|_{0,2}^2.
\]  
(4.6)

On the other hand, in the domain \( \{V_1 < V_2\} \), we have
\[
V_2 \leq V_1 \leq V_2 \leq V_1.
\]

Hence, we deduce that
\[
\int_0^T \chi_{\{V_1 < V_2\}} \left( k_{1,s}^- + k_{2,s}^+ \right) (V_2,s - V_1,s) \, ds \\
= \int_0^T \chi_{\{V_1 < V_2\}} \left[ (V_2,s - V_1,s) k_{1,s}^- - (V_1,s - V_1,s) k_{1,s}^- \right] \, ds \\
+ \int_0^T \chi_{\{V_1 < V_2\}} \left[ (V_2,s - V_2,s) k_{2,s}^+ - (V_1,s - V_2,s) k_{2,s}^+ \right] \, ds \\
\leq \int_0^T \chi_{\{V_1 < V_2\}} \left[ -(V_1,s - V_1,s) k_{1,s}^- + (V_2,s - V_2,s) k_{2,s}^+ \right] \, ds. 
\]  
(4.7)

Since \((V_1 - V_1) k_1^- \leq 0\) and \((V_2 - V_2) k_2^+ \geq 0\) a.e. in \( \Omega \times Q \), then the last two equalities in (3.5) imply that
\[
\int_0^T \chi_{\{V_1 < V_2\}} \left( V_1,s - V_1,s \right) k_{1,s}^- \, ds = \int_0^T \chi_{\{V_1 < V_2\}} \left( V_2,s - V_2,s \right) k_{2,s}^+ \, ds = 0. 
\]  
(4.8)

From (4.5)-(4.8), we conclude that
\[
\| (\Delta V)^- \|_{1,2}^2 + \| (\Delta V)^- \|_{0,2}^2 + \| \Delta Z \chi_{\{\Delta V \leq 0\}} \|_{0,2}^2 \\
\leq C(\kappa, K, T) \mathbb{E} \left( |(\Delta \varphi)|_{0,2}^2 + \int_0^T \int_{\mathbb{R}^n} (\Delta f_s)^- (\Delta V_s)^- \, dx \, ds + \| (\Delta V)^- \|_{0,2}^2 \right).
\]

Denote
\[
\|V\|_{k,2}^2 = E \left( \sup_{s \in [t,T]} |V_{s,k}^2| \right) \quad \text{and} \quad \|V\|_{k,2; t}^2 = E \left( \int_t^T |V_{s,k}^2| \, ds \right).
\]

Consider the above estimates only on the time interval \([t,T]\), then we have
\[
\| (\Delta V)^- \|_{1,2}^2 + \sup_{s \in [t,T]} \mathbb{E} \| (\Delta V)^- \|_{0,2}^2 + \| \Delta Z \chi_{\{\Delta V \leq 0\}} \|_{0,2}^2 \\
\leq \| (\Delta V)^- \|_{1,2}^2 + \| (\Delta V)^- \|_{0,2}^2 + \| \Delta Z \chi_{\{\Delta V \leq 0\}} \|_{0,2}^2 \\
\leq C(\kappa, K, T) \mathbb{E} \left( |(\Delta \varphi)|_{0,2}^2 + \int_t^T \int_{\mathbb{R}^n} (\Delta f_s)^- (\Delta V_s)^- \, dx \, ds + \int_t^T |(\Delta V)^-|_{0,2}^2 \right).
\]

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Applying the Gronwall inequality, we have (4.2). Repeating the similar argument, we can derive that
\[
\|(\Delta V)^+\|_{1,2}^2 + \|((\Delta V)^+)\|_{0,2}^2 + \|\Delta Z\chi_{\{\Delta V \geq 0}\}}\|_{0,2}^2
\leq C(\kappa, K, T) E \left( (\Delta \varphi)^+_{0,2}^2 + \int_0^T \int_{\mathbb{R}^n} (\Delta f_s)^+ (\Delta V_s)^+ \, dx \, ds \right).
\]
So, (4.3) is obvious. And (4.4) follows from the Hölder inequality. \hfill \Box

From Lemma 4.3, we can deduce the following comparison theorem for SBSPDVI:

**Theorem 4.4.** Let Assumptions (V1)-(V5) be satisfied. Moreover, \((V_i, Z_i, k^+_i, k^-_i) \in \mathbb{H}^{2,2} \times \mathbb{H}^{1,2} \times \mathbb{H}^{0,2} \times \mathbb{H}^{0,2}\) is the strong solution to SBSPDVI (3.1) with \((f, \varphi, \underline{V}, \underline{V}) \triangleq (f_i, \varphi_i, \underline{V}_i, \underline{V}_i)\) for \(i = 1, 2\). If \(f_1 \geq f_2, \varphi_1 \geq \varphi_2, \underline{V}_1 \geq \underline{V}_2\), and \(\underline{V}_1 \geq \underline{V}_2\), then \(V_1 \geq V_2\) a.e. in \(\Omega \times Q\).

**Remark 4.1.** In fact, applying the method in the proof of Theorem 4.9, we can loosen Assumptions (V3) and (V4) as (V3) and (V4). Moreover, we can remove the assumption \((V_i, Z_i, k^+_i, k^-_i) \in \mathbb{H}^{2,2} \times \mathbb{H}^{1,2} \times \mathbb{H}^{0,2} \times \mathbb{H}^{0,2}\).

In view of [27, Theorem 5.3], we have the following existence and uniqueness of the strong solution for a linear BSPDVI:

**Lemma 4.5.** Let Assumptions (V1), (V2) and (V4), (V5) be satisfied and \(F \equiv 0\). Then BSPDVI (3.1) has a unique strong solution \((V, Z, k^+, k^-) \in \mathbb{H}^{2,2} \times \mathbb{H}^{1,2} \times \mathbb{H}^{0,2} \times \mathbb{H}^{0,2}\) such that
\[
\|V\|_{1,2}^2 + \|V\|_{1,2}^2 + \|Z\|_{1,2}^2 + \|k^+\|_{0,2}^2 + \|k^-\|_{0,2}^2
\leq C(\kappa, K, T) \left( E \left[ \|\varphi\|_{1,2}^2 + \|f\|_{0,2}^2 + \|V\|_{1,2}^2 + \|\underline{V}\|_{1,2}^2 \right]
+ \|Z\|_{0,2}^2 + \|Z\|_{0,2}^2 + \|h\|_{0,2}^2 \right) \tag{4.9}
\]

In order to apply the Banach fixed point theorem, we need define the weak solution of SBSPDVI (3.1) as follows:

**Definition 4.1.** If \((V, Z, k^+, k^-) \in \mathbb{H}^{1,2} \times \mathbb{H}^{0,2} \times \mathbb{H}^{0,2} \times \mathbb{H}^{0,2}\) such that
\[
\int_{\mathbb{R}^n} V_t \eta \, dx = \int_{\mathbb{R}^n} \varphi \eta \, dx - \int_t^T \int_{\mathbb{R}^n} \left( a^{ij} D_i V + \sigma^{jk} Z^k \right) D_j \eta + \left( b^i - D_j a^{ij} \right) D_i \underline{V}
+ c \underline{V} + \left( \mu^k - D_i \sigma^{ik} \right) Z^k + f + F(\cdot, V, D V, Z) + k^+ - k^- \right) \eta \, dx \, ds
\]
\[
- \int_t^T \int_{\mathbb{R}^n} Z^k \eta \, dx \, dB^k \quad \text{for any } \eta \in H^{1,2}, \ t \in [0, T] \text{ and a.s. in } \Omega; \tag{4.10}
\]

\[
V \leq \underline{V}, \quad k^\pm \geq 0 \quad \text{a.e. in } \Omega \times Q;
\]

\[
\int_0^T (V_t - \underline{V}_t) \, dt = \int_0^T (\underline{V}_t - V_t) \, dt = 0 \quad \text{a.e. in } \Omega \times \mathbb{R}^n.
\]

Then \((V, Z, k^+, k^-)\) is called a weak solution of SBSPDVI (3.1).
The difference between strong solutions and weak solutions of SBSPDVI is only in the first equation, that is just the difference between strong solutions and weak solutions of BSPDEs. So, according to [10, Proposition 2.2] or the proof of [27, Theorem 5.3], we have the following proposition:

**Proposition 4.6.** Let Assumptions (V1)-(V3) be satisfied. If \((V,Z,k^+,k^-) \in H^{2,2} \times H^{1,2} \times H^{0,2} \times H^{0,2}\) is the strong solution of SBSPDVI (3.1), then it is also a weak solution. Moreover, if the weak solution of SBSPDVI (3.1) satisfies \(V \in H^{2,2}\) and \(Z \in H^{1,2}\), then it is also a strong solution.

Moreover, we have the following equivalence between the strong solution and the weak solution under Assumptions (V1)-(V5).

**Lemma 4.7.** Let Assumptions (V1)-(V5) be satisfied. \((V,Z,k^+,k^-) \in H^{2,2} \times H^{1,2} \times H^{0,2} \times H^{0,2}\) is the strong solution of SBSPDVI (3.1), then it is also a weak solution. Moreover, the weak solution of SBSPDVI (3.1) is also a strong solution.

**Proof.** According to Proposition 4.6, it is sufficient to prove that \(V \in H^{2,2}\) and \(Z \in H^{1,2}\) if \((V,Z,k^+,k^-)\) is the weak solution of SBSPDVI (3.1).

In view of the first equation in (4.10), we know that \((V,Z)\) is a weak solution of the following BSPDE:

\[
\begin{cases}
    dV_t = -(LV_t + M^k Z^k_t + g_t) dt + Z^k_t dB^k_t; \\
    V_T = \varphi
\end{cases}
\]

with

\[g(t) = f(t, \cdot, V_t, DZ_t, Z_t) + k^+_t - k^-_t.\]

Since \((V,Z,k^+,k^-) \in H^{1,2} \times H^{0,2} \times H^{0,2} \times H^{0,2}\), then Assumption (V5) implies that \(g \in H^{0,2}\).

According to [10, Corollary 3.4] or [27, Lemma 2.2], we deduce that BSPDE (4.11) has a unique strong solution such that \((V^*, Z^*) \in H^{2,2} \times H^{1,2}\), which is also a weak solution of BSPDE (4.11).

On the other hand, [23, Corollary 3.4] implies that the weak solution of BSPDE (4.11) is unique. So, \((V,Z) = (V^*, Z^*) \in H^{2,2} \times H^{1,2}\). And the result in this lemma is obvious. □

Thank to the above preparations, we can establish the existence and uniqueness of the solution of SBSPDVI (3.1).

**Theorem 4.8.** Let Assumptions (V1)-(V5) be satisfied. Then SBSPDVI (3.1) has a unique strong solution \((V,Z,k^+,k^-)\) such that

\[
\|V\|_{2,2} + \|V\|_{1,2} + \|Z\|_{1,2} + \|k^+\|_{0,2} + \|k^-\|_{0,2} \\
\leq C(\kappa, K, T) \left( E[|\varphi|_{1,2}] + \|f\|_{0,2} + \|V\|_{1,2} + \|\overline{V}\|_{1,2} + \|Z\|_{0,2} + \|\overline{Z}\|_{0,2} + \|h\|_{0,2} + \|F(\cdot, 0, 0, 0)\|_{0,2} \right).
\]

Moreover, Let the assumptions in Theorem 3.2 be satisfied. Then the strong solution of SBSPDVI (3.1) coincides with the value of Problem \(\mathcal{D}_{tx}\).
Proof. We will apply the Banach fixed point theorem to establish the existence of the strong solution of BSPDVI (3.1). The proof is divided into five steps:

**Step 1.** Construct the proper space and mapping for applying the Banach fixed point theorem. Let $D \equiv H^{1,2} \times H^{0,2}$ with the norm

$$\| (V, Z) \|_D \equiv \| V \|_{1,2} + \| Z \|_{0,2}.$$

Let the mapping $A$ be defined as follows. Given a function $(v, z) \in D$, then it is clear that

$$|F(\cdot, v, Dv, z)| \leq |F(\cdot, 0, 0, 0)| + K \left( |v| + |Dv| + |z| \right) \in H^{0,2}.$$

Hence, the following linear BSPDVI has a unique strong solution $(V, Z, k^+, k^-)$ and $(V, Z) \in D$.

$$
\begin{cases}
    dV_t = -(\mathcal{L}V_t + \mathcal{M}^k Z_t^k + F_t) dt + Z_t^k dB_t^k & \text{if } V_t < V < \nabla_t; \\
    dV_t \leq -(\mathcal{L}V_t + \mathcal{M}^k Z_t^k + F_t) dt + Z_t^k dB_t^k & \text{if } V_t = \nabla_t; \\
    dV_t \geq -(\mathcal{L}V_t + \mathcal{M}^k Z_t^k + F_t) dt + Z_t^k dB_t^k & \text{if } V_t = \overline{\nabla}_t; \\
    V_T = \varphi
\end{cases}
$$

(4.13)

with $F_t(x) \equiv F(t, x, v_t(x), Dv_t(x), z_t(x))$. According to Lemma 4.7, $(V, Z, K^+, K^-)$ is also the unique weak solution of BSPDVI (4.13). Define $A$ by

$$A(v, z) \to (V, Z), \quad (v, z) \in D.$$

**Step 2.** Testify that the mapping $A$ is a strict contraction provided $T$ is small enough. That means there exists a $\varsigma \in (0, 1)$ such that

$$\| (\Delta V, \Delta Z) \|_D \leq \varsigma \| (\Delta v, \Delta z) \|_D,$$

where $\Delta V = V_1 - V_2$, $\Delta Z = Z_1 - Z_2$, $\Delta v = v_1 - v_2$, $\Delta z = z_1 - z_2$.

In fact, from (4.3), we deduce that

$$\| \Delta V \|_{1,2}^2 + \| \Delta V \|_{0,2}^2 + \| \Delta Z \|_{0,2}^2$$

$$\leq C \mathbb{E} \int_0^T \int_{\mathbb{R}^n} \left[ \left( F(\cdot, v_1, Dv_1, z_1) - F(\cdot, v_2, Dv_2, z_2) \right) \Delta V \right] ^+ dx \, ds$$

$$\leq C \mathbb{E} \int_0^T \int_{\mathbb{R}^n} \left( |\Delta v| + |D\Delta v| + |\Delta z| \right) |\Delta V| \, dx \, ds$$

$$\leq C \mathbb{E} \int_0^T \left( |\Delta v|_{1,2} + |\Delta z|_{0,2} \right) |\Delta V|_{0,2} \, ds$$

$$\leq C \sqrt{T} \left( \| \Delta v \|_{1,2} + \| \Delta z \|_{0,2} \right) \| \Delta V \|_{0,2} \leq \| \Delta V \|_{0,2} \leq CT \left( \| \Delta v \|_{1,2}^2 + \| \Delta z \|_{0,2}^2 \right).$$

Hence, if $T$ is small enough such that $CT < 1/2$, then we have

$$\| (\Delta V, \Delta Z) \|_D \leq \frac{1}{2} \| (\Delta v, \Delta z) \|_D.$$
Step 3. We prove that SBSPDVI (3.1) has a unique strong solution provided $CT < 1/2$.

Fix any point $(V_0, Z_0) \in \mathcal{D}$ and thereafter iteratively define $(V_{m+1}, Z_{m+1}) = A(V_m, Z_m)$ for $m = 0, 1, \cdots$. Applying the Banach fixed point theorem, we deduce that $\{(V_{m+1}, Z_{m+1})\}$ is a cauchy sequence in the space $\mathcal{D}$. Then there exists a $(V, Z) \in \mathcal{D}$ such that

$$\|(V_m - V, Z_m - Z)\|_\mathcal{D} \to 0 \text{ as } m \to +\infty.$$  

Denoted by $(V_m, Z_m, k^+_m, k^-_m)$ the strong solution of BSDPVI (4.1) with $(v, z) \triangleq (V_{m-1}, Z_{m-1})$. Then Lemma 4.5 implies that

$$\|V_m\|_{L^2} + \|Z_m\|_{L^2} + \|k^+_m\|_{L^2} + \|k^-_m\|_{L^2} \leq C(k, K, T) \left( \mathbb{E}[\|\varphi\|_{L^2} + \|f\|_{L^2} + \|V\|_{L^2} + \|Z\|_{L^2} + \|h\|_{L^2} + \|F_{m-1}\|_{L^2} \right)$$

with $F_{m-1} \triangleq F(\cdot, V_{m-1}, DV_{m-1}, Z_{m-1})$ and

$$
\|F_{m-1}\|_{L^2} \leq C \left( \|F(\cdot, 0, 0, 0)\|_{L^2} + \|V_{m-1}\|_{L^2} + \|Z_{m-1}\|_{L^2} \right) \leq C.
$$

So, there exist $k^+, k^-$ and a subsequence of $\{(V_m, Z_m, k^+_m, k^-_m)\}$, still denoted by itself, such that as $m \to \infty$,

$$V_m \to V \text{ in } H^{1,2}, \quad Z_m \to Z \text{ in } H^{0,2}, \quad k^+_m \to k^+, \quad k^-_m \to k^- \text{ weakly in } H^{0,2}.$$ 

Moreover, we compute

$$
\|F_m - F\|_{L^2} \leq C \left( \|V_m - V\|_{L^2} + \|Z_m - Z\|_{L^2} \right) \to 0 \text{ as } m \to \infty.
$$

So, it is not difficult to deduce that $(V, Z, k^+, k^-)$ is a weak solution of SBSPDVI (3.1) by the method in the proof of [27, Theorem 5.3]. In view of Theorem 4.4 and Lemma 4.7, we derive that $(V, Z, k^+, k^-)$ is also the unique strong solution of SBSPDVI (3.1).

Step 4. Finally, we prove that SBSPDVI (3.1) has a unique strong solution for any given $T > 0$.

We select $T_1 > 0$ so small such that $CT_1 \leq 1/2$. We can find a strong solution existing on time interval $[T - T_1, T]$.

Since $V_t \in L^{1,2}$ a.e. in $[0, T]$, upon redefining $T_1$ if necessary, we may assume $V_{T-T_1} \in L^{1,2}$. We can then repeat the argument above to extend our solution to the time interval $[T - 2T_1, T - T_1]$. Continuing, after finitely many steps we can prove the strong solution existing on the full interval $[0, T]$.

The uniqueness of the solution for SBSPDVI (3.1) comes from Theorem 4.4.

Step 5. Prove the estimate (4.12). Assume $(V_1, Z_1, k^+_1, k^-_1)$ is the strong solution of SBSPDVI (3.1) with $F \equiv 0$. Then we can rewrite the first equation in the definition 3.1 as the following

$$V_{1,t} = \varphi + \int_t^T (LV_{1,s} + \mathcal{M}^k Z^k_{1,s} + f_{1,s} + F(s, V_{1,s}, DV_{1,s}, Z_{1,s}) + k^+_1 - k^-_1) ds - \int_t^T Z^k_{1,s} dB^k_s$$

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with \( f_1 = f - F(\cdot, V_1, DV_1, Z_1) \). According to the estimate (4.4) in Lemma 4.3 and Lemma 4.5, we have

\[
\|V\|_{1,2} + \|Z\|_{0,2} \leq \left( \|\Delta V_1\|_{1,2} + \|\Delta Z_1\|_{0,2} \right) + \left( \|V_1\|_{1,2} + \|Z_1\|_{0,2} \right)
\]

\[
\leq C(\kappa, K, T) \|F(\cdot, V_1, DV_1, Z_1)\|_{0,2} + \left( \|V_1\|_{1,2} + \|Z_1\|_{0,2} \right)
\]

\[
\leq C(\kappa, K, T) \left( \|F(\cdot, 0, 0, 0)\|_{0,2} + \|V_1\|_{1,2} + \|Z_1\|_{0,2} \right) \leq C(\kappa, K, T) L
\]

with \( \Delta V_1 = V - V_1, \Delta Z_1 = Z - Z_1 \), and

\[
L = \mathbb{E}[\|\varphi\|_{1,2} + \|f\|_{0,2} + \|V\|_{1,2} + \|Z\|_{0,2} + \|\nabla\|_{1,2} + \|\mathcal{V}\|_{0,2} + \|h\|_{0,2} + \|F(\cdot, 0, 0, 0)\|_{0,2}]
\]

In view of Lemma 4.5, we have

\[
\|V\|_{2,2} + \|V\|_{1,2} + \|Z\|_{1,2} + \|k^+\|_{0,2} + \|k^-\|_{0,2}
\]

\[
\leq C(\kappa, K, T) \left( L + \|F(\cdot, V, DV, Z)\|_{0,2} \right)
\]

\[
\leq C(\kappa, K, T) \left( L + \|F(\cdot, 0, 0, 0)\|_{0,2} + \|V\|_{1,2} + \|Z\|_{0,2} \right) \leq C(\kappa, K, T) L.
\]

So, we have proved the estimate (4.12). \( \square \)

In fact, the assumptions imposed on the payoff function in Theorem 4.8 is too strong, and many financial models don’t satisfy them. For example, if let \( X_t \) be the logarithmic function of the stock’s price, then the terminal payoff of the American call option is \( (e^x - K)^+ \), where \( K \) is the strike price. In this case, \( (e^x - K)^+ \notin \mathbb{H}^{1,2} \). But based on Theorem 4.8, we can loosen the assumptions, which are proper for most of the financial models.

**Theorem 4.9.** Let Assumptions (V1)-(V2) and (V5) be satisfied. Assume that there exists a nonnegative constant \( \lambda \) such that Assumptions (V3)\(_\lambda\) and (V4)\(_\lambda\) are satisfied. Then SBSPDVI (3.1) has a unique strong solution \((V, Z, k^+, k^-)\) such that

\[
\|V\|_{2,2,\lambda} + \|V\|_{1,2,\lambda} + \|Z\|_{1,2,\lambda} + \|k^+\|_{0,2,\lambda} + \|k^-\|_{0,2,\lambda} \leq \left( \mathbb{E}[\|\varphi\|_{1,2,\lambda} + \|f\|_{0,2,\lambda} + \|V\|_{1,2,\lambda} + \|\nabla\|_{1,2,\lambda}
\]

\[
+ \|Z\|_{0,2,\lambda} + \|h\|_{0,2,\lambda} + \|F(\cdot, 0, 0, 0)\|_{0,2,\lambda} \right).
\]

Moreover, Let \( X \) be the solution of SDE (2.3), and \( e^{-\lambda|x|^2}V(X), e^{-\lambda|x|^2}\varphi(X_T) \in \mathcal{S}^2 \) and \( e^{-\lambda|x|^2}\varphi(X_T) \in \mathcal{S}^2 \). Assume that Assumptions (D1), (D2) and (D3)\(_\lambda\) are satisfied and the equations (3.7) and (3.8) hold. Then \( V(t, x) \) is the value of Problem \( P_{tx} \), and \( (\tau_1^*, \tau_2^*) \) defined in Theorem 3.2 is a Nash equilibrium point of Problem \( P_{tx} \).

**Proof.** In the following, we apply a proper transformation to SBSPDVI (3.1) so that the new problem satisfies the assumptions in Theorem 4.8. Denote

\[
\psi(x) \triangleq e^{\lambda|x|} - \lambda|x|.
\]

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It is not difficult to check that $\psi \in C^2$ and

$$D_t \psi = \frac{\lambda x_i}{|x|}(e^{\lambda |x|} - 1), \quad \left| \frac{D \psi}{\psi} \right| \leq \frac{\lambda e^{\lambda |x|}}{e^{\lambda |x|} - \lambda |x|} \leq \frac{\lambda}{1 - e^{-1}}.$$ 

Denote

$$v = V / \psi, \quad z = Z / \psi, \quad \bar{v} = \bar{V} / \psi, \quad \bar{v} = \bar{V} / \psi, \quad \bar{f} = f / \psi, \quad \bar{\phi} = \varphi / \psi.$$ 

Then SBSPDVI (3.1) is equivalent to the following SBSPDVI

$$\begin{align*}
d v_t &= -(\bar{\mathcal{L}} v_t + \tilde{\mathcal{M}} k z_t^k + \bar{f}_t + \bar{F}(t, \cdot, v_t, D v_t, z_t)) \, dt + z_t^k \, dB_t^k \quad \text{if } v_t < v_T < \bar{v}_t; \\
d v_t &\leq -(\bar{\mathcal{L}} v_t + \tilde{\mathcal{M}} k z_t^k + \bar{f}_t + \bar{F}(t, \cdot, v_t, D v_t, z_t)) \, dt + z_t^k \, dB_t^k \quad \text{if } v_t = \bar{v}_t; \\
d v_t &\geq -(\bar{\mathcal{L}} v_t + \tilde{\mathcal{M}} k z_t^k + \bar{f}_t + \bar{F}(t, \cdot, v_t, D v_t, z_t)) \, dt + z_t^k \, dB_t^k \quad \text{if } v_t = v_T; \\
v_T &= \bar{\phi},
\end{align*}$$

(4.15)

where

$$\bar{\mathcal{L}} v = \mathcal{L} v + \frac{1}{\psi} \left( 2a^{ij} D_i \psi D_j v + a^{ij} D_{ij} \psi^2 v + b^i D_i \psi v \right), \quad \tilde{\mathcal{M}} k z = \mathcal{M} k z + \sigma^{ik} \frac{D \psi}{\psi} z,$$

$$\bar{F}(t, x, v, y, z) = \frac{1}{\psi(x)} F(t, x, \psi(x)v, \psi(x)y + D \psi(x)v, \psi(x)z).$$

It is not difficult to check that SBSPDVI (4.15) satisfies all assumptions in Theorem 4.8. So, SBSPDVI (4.15) has a unique strong solution $(v, z, \bar{k}^+, \bar{k}^-)$ satisfying the corresponding estimate. Hence, SBSPDVI (3.1) has a unique strong solution $(V, Z, k^+, k^-)$ satisfying (4.14). Repeating an argument similar to the above and recalling Theorem 3.2, we can deduce the remain results in this theorem.

\[\square\]

5 The optimal stopping time problem as an extreme case of a Dynkin’s game.

In this section, we consider an optimal stopping time problem (denoted by Problem $\mathcal{O}$ hereafter), which involves only one choice variable of stopping times. We show that Problem $\mathcal{O}$ is a special case of Dynkin games under suitable conditions and identify the corresponding results about Problem $\mathcal{O}$ and BSPDVI with one obstacle.

The state $X$ is governed by SDE (2.3). The payoff is defined by

$$P_t(x; \tau) = \int_t^\tau G(u, X^{t,x}_u, P^{t,x}_u, Y_u) \, du - \int_t^\tau Y_u^{1,t} dW_u^1 - \int_t^\tau Y_u^{2,k} dB_u^k + \mathbb{E}_{\tau}(X^{t,x}_\tau) \chi(\tau < T) + \varphi(X^{t,x}_\tau) \chi(\tau = T), \quad \tau \in \mathcal{U}_{t,T}. \quad (5.1)$$

where $P_t(x; \tau)$ is just $P^{t,x}_t$, the diffusion term $Y = (Y^1, Y^2)$. They are similar to those in Dynkin game problem $\mathcal{O}_{t,x}$. The coefficients satisfy Assumption (D1) and (D2).
running payoff $G$ satisfies Assumption (D3), the terminal payoffs $V$ and $\varphi$ are similar to those in Dynkin game problem $\mathcal{O}_{tx}$, satisfying some proper assumptions.

The optimal stopping problem $\mathcal{O}_{tx}$, associated to the initial data $(t, x)$, is to find a stopping time $\tau^* \in \mathcal{U}_{t,T}$ such that

$$P_t(x; \tau^*) = V_t(x) \triangleq \text{ess} \sup_{\tau \in \mathcal{U}_{t,T}} P_t(x; \tau).$$

The random variable $V(t, x)$ is called the value of Problem $\mathcal{O}_{tx}$.

The HJB equation for Problem $\mathcal{O}_{tx}$ is the following BSPDVI with one obstacle:

$$
\begin{align*}
    dV_t &= -\left(\mathcal{L}V_t + \mathcal{M}^k Z_t^k + f_t + F(t, \cdot, V_t, DV_t, Z_t)\right) dt + Z_t^k dB_t^k & \text{if } V_t > \underline{V}_t; \\
    dV_t &\leq -\left(\mathcal{L}V_t + \mathcal{M}^k Z_t^k + f_t + F(t, \cdot, V_t, DV_t, Z_t)\right) dt + Z_t^k dB_t^k & \text{if } V_t = \underline{V}_t; \\
    V_T(x) &= \varphi(x),
\end{align*}
$$

(5.2)

where the operators $\mathcal{L}$ and $\mathcal{M}$ are defined by (3.2).

**Definition 5.1.** A triplet $(V, Z, k^+) \in \mathbb{H}^{2,2}_\lambda \times \mathbb{H}^{1,2}_\lambda \times \mathbb{H}^{0,2}_\lambda$ is called a strong solution of BSPDVI (5.2) if it satisfies the following:

$$
\begin{align*}
    &V_t = \varphi + \int_t^T (\mathcal{L}V_s + \mathcal{M}^k Z_s^k + f_s + F(s, \cdot, V_s, DV_s, Z_s) + k^+_s) ds - \int_t^T Z_s^k dB_s^k \\
    &V \geq \underline{V}_t, \quad k^+ \geq 0 \quad \text{a.e. in } \Omega \times \overline{Q}; \\
    &\int_0^T (V_t - \underline{V}_t) k^+_t dt = 0 \quad \text{a.e. in } \Omega \times \mathbb{R}^n.
\end{align*}
$$

Next, we will consider the following assumptions:

**Assumption (O1)$_\lambda$ (Regularity).** $f \in \mathbb{H}^{0,2}_\lambda, \varphi \in \mathbb{L}^{1,2}_\lambda, \underline{V} \in \mathbb{H}^{1,2}_\lambda$ with some nonnegative number $\lambda$. The lower obstacle $\underline{V}$ is a continuous semimartigales of the following form

$$d \underline{V}_t = -g_t dt + Z_t^k dB_t^k$$

with some $(Z, g) \in \mathbb{H}^{0,2}_\lambda \times \mathbb{H}^{0,2}_\lambda$. Moreover, there exist two stochastic fields $\widehat{V} \in \mathbb{H}^{2,2}_\lambda \cap S^{1,2}_\lambda, \widehat{h} \in \mathbb{H}^{0,2}_\lambda$ such that

$$\widehat{V} \geq \underline{V} \quad \text{and} \quad d\widehat{V}_t = -\widehat{g}_t dt + \widehat{Z}_t^k dB_t^k$$

with some $(\widehat{Z}, \widehat{g}) \in \mathbb{H}^{1,2}_\lambda \times \mathbb{H}^{0,2}_\lambda$. And $\mathcal{L} \underline{V} + \mathcal{M}^k Z^k - g + h \geq 0$ hold in the sense of distribution.

**Assumption (O2) (Compatibility).** $\underline{V}_T \leq \varphi$.

The assumption notation (O1)$_0$ will be abbreviated as (O1) if there is no confusion.

Identical to the proof of Lemma 4.3 and Theorem 4.9, we have the following comparison theorem:
\textbf{Theorem 5.1.} Let Assumptions (V1), (V2) and (V3) and Assumptions (O1) and \((O2)\) be satisfied. Let \((V_i, Z_i, k_i^+)\) be the strong solution of BSPDVI \((5.1)\) associated with \((f_i, \varphi_i, V_i)\) for \(i = 1, 2\). If \(f_1 \geq f_2, \varphi_1 \geq \varphi_2\), and \(V_1 \geq V_2\), then \(V_1 \geq V_2\) a.e. in \(\Omega \times Q\).

The following lemma gives the relationship between Problems \(\mathcal{D}_{tx}\) and \(\mathcal{O}_{tx}\), and between BSPDVIs \((3.1)\) and \((5.1)\).

\textbf{Theorem 5.2.} Let Assumptions \((D1)-(D3)\) (resp. \((V1)-(V3)\)), \((O1)\) and \((O2)\) be satisfied. Then there exists a stochastic field \(\mathbf{V}\) such that Assumptions \((V3)'\) and \((V4)\) are satisfied. Moreover, Problems \(\mathcal{O}_{tx}\) and \(\mathcal{D}_{tx}\) (resp. BSPDVIs \((5.1)\) and \((3.1)\)) are equivalent to each other. And we have the following estimate:

\[
\|\mathbf{V}\|_{2,2} + \|\mathbf{Z}\|_{1,2} + \|\mathbf{\varphi}\|_{0,2} \leq C(\kappa, K, T) \left( \mathbb{E} [\|\varphi\|_{1,2}] + \|G(\cdot, 0)\|_{0,2} + \|f\|_{0,2} + \|F(\cdot, 0)\|_{0,2} + \|\mathbf{V}\|_{2,2} + \|\mathbf{Z}\|_{1,2} + \|\mathbf{\varphi}\|_{0,2} + 1 \right). (5.3)
\]

\textbf{Proof.} Suppose that Assumptions \((D1)-(D3)\), \((O1)\), and \((O2)\) are satisfied. Next, we construct an upper obstacle \(\mathbf{V}\) in Problem \(\mathcal{D}_{tx}\).

Let \((\mathbf{V}, \mathbf{Z})\) be the strong solution of the following BSPDE:

\[
\begin{cases}
        d\hat{V}_t = -\left[ L \hat{V}_t + M^k \hat{Z}_t^k + G(t, \cdot, \hat{V}_t, N\hat{V}_t, \hat{Z}_t + M\hat{V}_t) + \hat{f}_t \right] dt + \hat{Z}_t^k dB_t^k; \\
        \hat{V}_T = \max \left\{ 0, \varphi, \hat{V}_T \right\},
\end{cases}
\]

where \(L \) and \(M \) are defined by \((3.6)\) and \(\hat{f} \approx \max \left\{ 0, -G(\cdot, 0, 0, 0), \hat{f} \right\} \in \mathbb{H}^{0,2}, \hat{f} \approx \mathbf{g} - L \hat{V} - M^k \hat{Z}_t^k - G(\cdot, \hat{V}, N\hat{V}, \hat{Z} + M\hat{V}).\)

According to Lemma 4.1, BSPDE \((5.4)\) has a strong solution \((\mathbf{V}, \mathbf{Z}) \in \mathbb{H}^{2,2} \times \mathbb{H}^{1,2}.\) Moreover, the comparison theorem for BSPDEs in \([10]\) implies that \(\hat{V} \geq 0.\)

Since \(\hat{V}\) satisfies

\[
\begin{cases}
        d\hat{V}_t = -\left[ L \hat{V}_t + M^k \hat{Z}_t^k + G(t, \cdot, \hat{V}_t, N\hat{V}_t, \hat{Z}_t + M\hat{V}_t) + \hat{f}_t \right] dt + \hat{Z}_t^k dB_t^k, \\
        \hat{V}_T \leq \hat{V}_T,
\end{cases}
\]

Then the comparison theorem for BSPDEs in \([10]\) implies that \(\hat{V} \geq \hat{V} \geq \underline{V} .\)

Define \(\mathbf{\varphi} \equiv \hat{V} + (1 + |x|^{n+1})^{-1}.\)

Then \(\mathbf{\varphi} \in \mathbb{H}^{2,2}, \mathbf{\varphi} > \underline{V}, \mathbf{\varphi} > \varphi > \underline{V}_T,\) and

\[
d\mathbf{\varphi}_t = -\mathbf{g}_t dt + Z^k_t dB_t^k, \quad \mathbf{g} \equiv L \hat{V} + M^k \hat{Z}_t^k + G(t, \cdot, \hat{V}_t, N\hat{V}_t, \hat{Z}_t + M\hat{V}_t) + \hat{f}.
\]

It is clear that \(\mathbf{\varphi} \in \mathbb{H}^{0,2}\) and \(\mathbf{Z} = \mathbf{Z} \in \mathbb{H}^{1,2}.\) Hence, \(\underline{V}, \mathbf{V}, \mathbf{\varphi}\) satisfy Assumptions \((V3)'\) and \((V4)\). The estimate \((5.3)\) follows from Lemma 4.1.
In the following we prove that Problems $\mathcal{O}_{tx}$ and $\mathcal{D}_{tx}$ are equivalent to each other. We first claim
\[ R_t(x; \tau_1, \tau_2) \geq P_t(x; \tau_1) \text{ for any } \tau_1, \tau_2 \in \mathcal{U}_{t,T}. \tag{5.5} \]
Define
\[ R^{t,x}_{\tau_1, \tau_2} \triangleq \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] + \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] + \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right]. \]

Applying Theorem 3.1 and repeating the method in the proof of Theorem 3.2, we deduce that
\[ \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] + \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] = \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] + \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] \]
\[ \geq \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] + \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] \mathbb{E}_{\tau_1} \left[ X^{t,x}_{\tau_1, \tau_2} \right] \]
with
\[ M_2 \triangleq \int_{\tau_1}^{\tau_2} (N^2 v^2) (X^{t,x}_{\tau_1, \tau_2}) dW_t + \int_{\tau_1}^{\tau_2} (\tilde{Z}^k u + \tilde{M}^k u)(X^{t,x}_{\tau_1, \tau_2}) dB_t. \]
According to the comparison principle for BSDE, we can deduce that
\[ P_{\tau_1, \tau_2}^{t,x} \leq \mathbb{E}_{\tau_1, \tau_2} \left[ X^{t,x}_{\tau_1, \tau_2} \right] \]
Hence, we derive that
\[ P_{\tau_1, \tau_2}^{t,x} \leq R^{t,x}_{\tau_1, \tau_2}. \]
Again applying the comparison principle for BSDE, we can achieve (5.5).

If Problem $\mathcal{D}_{tx}$ has a saddle point $(\tau_1^*, \tau_2^*)$, then we have
\[ P_t(x; \tau_1^*) = R_t(x; \tau_1^*, T) \geq R_t(x; \tau_1^*, \tau_2^*) \geq R_t(x; \tau_1^*, \tau_2^*) \geq P_t(x; \tau_1), \]
where $\tau_1$ is an arbitrary stopping time in $\mathcal{U}_{t,T}$ and we have used (5.5) in the last inequality. Hence, Problem $\mathcal{D}_{tx}$ has an optimal stopping time $\tau_1^* \in \mathcal{U}_{t,T}$.

Suppose that Problem $\mathcal{O}_{tx}$ has an optimal stopping time $\tau_1^* \in \mathcal{U}_{t,T}$. Then we choose $\tau_2^* = T$. We see that $R_t(x; \tau_1^*, \tau_2^*) = P_t(x; \tau_1)$ for any $\tau_1 \in \mathcal{U}_{t,T}$. Hence, we have
\[ R_t(x; \tau_1^*, \tau_2^*) = P_t(x; \tau_1^*) \geq P_t(x; \tau_1) = R_t(x; \tau_1^*, \tau_2^*). \]
On the other hand, according to (5.5), we have
\[ R_t(x; \tau_1^*, \tau_2) \geq P_t(x; \tau_1^*) = R_t(x; \tau_1^*, \tau_2^*) \text{ for any } \tau_2 \in \mathcal{U}_{t,T}. \]
Hence, $(\tau_1^*, \tau_2^*)$ is a saddle point of Problem $\mathcal{D}_{tx}$. Until now we have proved that Problems $\mathcal{O}_{tx}$ and $\mathcal{D}_{tx}$ are equivalent.
Suppose now that Assumptions (V1)-(V3), (O1) and (O2) are satisfied. Denote by \((\tilde{V}, \tilde{Z})\) the solution of the following BSPDE:
\[
\begin{aligned}
\begin{cases}
d\tilde{V}_t = -[\mathcal{L} \tilde{V}_t + \mathcal{M}^k \tilde{Z}_t^k + F(t, \cdot, \tilde{V}_t, D\tilde{V}_t, \tilde{Z}_t) + \tilde{f}_t] dt + \tilde{Z}_t^k dB_t^k; \\
\tilde{V}_T = \max\{0, \varphi, \tilde{V}_T\},
\end{cases}
\end{aligned}
\]  
\tag{5.6}
where \(\mathcal{L}\) and \(\mathcal{M}\) are defined in (3.2) and \(\tilde{f}\) is defined as
\[
\tilde{f} \equiv \max\{f, 0, -F(\cdot, 0, 0, 0), \tilde{g} - \mathcal{L} \tilde{V} - \mathcal{M}^k \tilde{Z}^k - F(t, \cdot, \tilde{V}_t, D\tilde{V}_t, \tilde{Z}_t)\}.
\]
Moreover, we define
\[
\nabla \equiv \tilde{V} + (1 + |x|^{n+1})^{-1}.
\]

Repeating the above argument, we derive that BSPDE (5.6) has a strong solution \((\tilde{V}, \tilde{Z}) \in H^{2,2} \times H^{1,2}\). Moreover, we have \(\tilde{V} \geq V^+, \tilde{V} > V\), and \(V, \tilde{V}\) and \(\varphi\) satisfy Assumptions (V4) and (V5). The estimate (5.3) follows from Lemma 4.1.

So, SBSPDVI (3.1) has a unique strong solution \((V, Z, k^+, k^-)\) by Theorem 4.8.

On the other hand, since \(V \leq \tilde{V} < V\) and \((\tilde{V}, \tilde{Z})\) is a strong solution of BSPDE (5.6), then \((\tilde{V}, \tilde{Z}, 0, 0)\) is a strong solution of the following BSPDVI:
\[
\begin{aligned}
\begin{cases}
d\tilde{V}_t = -[\mathcal{L} \tilde{V}_t + \mathcal{M}^k \tilde{Z}_t^k + F(t, \cdot, \tilde{V}_t, D\tilde{V}_t, \tilde{Z}_t) + \tilde{f}_t] dt + \tilde{Z}_t^k dB_t^k \quad \text{if } \tilde{V}_t < \tilde{V}_t < \nabla_t; \\
d\tilde{V}_t \leq -[\mathcal{L} \tilde{V}_t + \mathcal{M}^k \tilde{Z}_t^k + F(t, \cdot, \tilde{V}_t, D\tilde{V}_t, \tilde{Z}_t) + \tilde{f}_t] dt + \tilde{Z}_t^k dB_t^k \quad \text{if } \tilde{V}_t = \nabla_t; \\
d\tilde{V}_t \geq -[\mathcal{L} \tilde{V}_t + \mathcal{M}^k \tilde{Z}_t^k + F(t, \cdot, \tilde{V}_t, D\tilde{V}_t, \tilde{Z}_t) + \tilde{f}_t] dt + \tilde{Z}_t^k dB_t^k \quad \text{if } \tilde{V}_t = \nabla_t; \\
\tilde{V}_T = \max\{0, \varphi, \tilde{V}_T\}.
\end{cases}
\end{aligned}
\]
In view of Theorem 4.4, \(\tilde{V} \geq V\) and \(\nabla > V\). So, we deduce that \(k^- = 0\) a.e. in \(\Omega \times Q\) and \((V, Z, k^+)\) is a strong solution of BSPDVI (5.2).

On the other hand, in view of Theorem 5.1, the strong solution of BSBDVI (5.2) is unique. So, the unique strong solutions of BSPDVI (5.2) and (3.1) coincide. \(\square\)

Repeating the argument in the proof of Theorem 4.9, we have the following theorem:

**Theorem 5.3.** Let Assumptions (V1)-(V2), (V3)\(_\lambda\), (O1)\(_\lambda\), and (O2) be satisfied. Then BSPDVI (5.2) has a unique strong solution \((V, Z, k^+)\) such that
\[
\|V\|_{2,2,\lambda} + \|\nu\|_{1,2,\lambda} + \|Z\|_{1,2,\lambda} + \|k^+\|_{0,2,\lambda}
\leq C(\kappa, K, T, \lambda) \left( \mathbb{E} \left[ \|\varphi\|_{1,2,\lambda} + \|F(\cdot, 0, 0, 0)\|_{0,2,\lambda} + \|f\|_{0,2,\lambda} + \|\nabla\|_{1,2,\lambda} + \|Z\|_{0,2,\lambda} \\
+ \|g\|_{0,2,\lambda} + \|h\|_{0,2,\lambda} + \|\tilde{V}\|_{2,2,\lambda} + \|\tilde{Z}\|_{1,2,\lambda} + \|\tilde{g}\|_{0,2,\lambda} + 1 \right] \right).
\]
Moreover, Let \(X\) be a solution of SDE (2.3), and \(Y(X), \tilde{V}(X) \in S^2\) and \(\varphi(X_T) \in L^2\). Assume that Assumptions (D1)-(D2), (D3)\(_\lambda\) are satisfied, and equations (3.7) and (3.8) hold. Then \(V(t, x)\) is the value of Problem \(\tilde{\sigma}\), and its optimal stopping time \(\tau^*\) can be described as
\[
\tau^* \equiv \inf\{s \in [t, T] : V_s(X_s) = \underline{V}_s(X_s)\} \wedge T.
\]
6 Two examples.

In this section, we give two financial examples.

**Example 6.1. The convertible bond pricing model**

Suppose that the underlying asset of the convertible bond is the price $S$ of the issuer’s stock, which is governed by the following SDE:

$$
\begin{align*}
S_t &= S + \int_t^s \tilde{r}_u(S_u)S_u du + \int_t^s \tilde{\gamma}(u, S_u) S_u dW_u, \\
\tilde{r}_s(S) &= r(S) + \int_t^s \alpha_u(\tilde{r}_u(S)) du + \int_t^s \tilde{\gamma}_u(\tilde{r}_u(S)) dB_t,
\end{align*}
$$

where $W$ and $B$ are one-dimensional independent standard Brownian motions. $\tilde{\gamma}$ is a deterministic function of $(u, S)$, and represents the volatility of the return on the stock. $\tilde{r}_u$ is the appreciation rate of the stock, which is a Jacobi stochastic process (a mean-reverting diffusion) with upper and low bounds for any fixed $S$, and governed by the second SDE (for example, [8, 20]).

Then under proper assumptions, the stochastic process $X = \ln S$ can be described as (2.3) with $n = k = l = 1$, and the parameters in (2.3) are as follows:

$$
x = \ln S, \quad \beta_t = r_t - \frac{\gamma^2_t}{2}, \quad r_t(w, x) \Delta \tilde{r}_u(w, e^x), \quad \gamma_t(x) \Delta \tilde{\gamma}(t, e^x), \quad \theta_t \equiv 0.
$$

Moreover, if we imposed some proper assumptions on the coefficient $\tilde{\gamma}$, $\alpha$, $\tilde{\gamma}$, then the coefficient functions $\beta$, $\gamma$ satisfy Assumptions (D1) and (D2).

The issuer of the convertible bond issues the bond to raise their capital, and the bondholder of the bond buys the bond to gain the return.

The bondholder buys a share of convertible bond from the issuer, then he continuously receives coupon paid from the issuer at a constant rate $\rho$ before maturity $T$ and converting the bond. Prior to maturity, the bondholder has the right to convert the bond into the firm’s stock with the constant conversion factor $\rho \in (0, 1)$, then gets $\rho S$ stock gain from the firm after converting. Moreover, the firm has the right to call the bond and force the bondholder to either surrender it to the firm for a previously agreed price $K$ or convert it into the stock with the conversion factor $\rho$. If neither the bondholder nor the firm exercises their right before maturity, the bondholder must sell the bond to the firm at a preset value $L \leq K$ or convert it into the firm’s stock at expiry date $T$. So, the bondholder receives $\max\{L, \rho S\}$ from the firm at maturity.

So, in the life of the convertible bond, the discounted value of the overall payoff of the bondholder can be described as the following:

$$
R(x, t; \tau_1, \tau_2) = \int_t^{\tau_1 \wedge \tau_2} \rho D_u^t du + K D_{\tau_2}^t I_{\tau_2 < \tau_1} + \rho \exp\left( X_{\tau_1}^{t, x} \right) D_{\tau_1}^t I_{\tau_1 \leq \tau_2, \tau_1 < T} + \max\left\{ L, \rho \exp\left( X_{T}^{t, x} \right) \right\} D_{\tau_2}^T I_{\tau_1 = \tau_2 = T}, \quad D_s^t = \exp\left\{ - \int_t^s c_u du \right\},
$$

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where $D^*_s$ is the discounted process, and $c_u$ is the discounted rate process, bounded and $\mathcal{P}^B$-measurable. And $(t,x)$ means that logarithmic value of the stock’s price is equal to $x$ at time $t$, and $\tau_1, \tau_2 \in \mathcal{U}_{t,T}$ represents the bondholder’s conversion strategy and the issuer’s calling strategy, respectively.

Since the bondholder’s gain comes from the issuer. Then the discounted value of the issuer’s overall cost is $R(x,t;\tau_1,\tau_2)$, too.

Now, we consider a model of ambiguity aversion. As we all know, ambiguity (or Knightian uncertainty) can be represented as a g-evaluation in a general context (see [3]).

Fix a positive parameter $\kappa$ and take a set, $\Theta$, of probability measures such that (see [4, 24])

$$\Theta = \left\{ Q^\xi : \frac{dQ^\xi}{dP} = \exp \left( -\frac{1}{2} \int_0^T |\zeta_s|^2 ds - \int_0^T \zeta_{1,s} dW_s - \int_0^T \zeta_{2,s} dB_s \right), \ |\zeta| \leq \kappa \right\},$$

where $\zeta = (\zeta_1, \zeta_2)$.

Set $\Theta$ represents the $\kappa$–ignorance measure (see [4]). The worse-case expectation $\min_{Q \in \Theta} \mathbb{E}[\xi|\mathcal{F}_t]$ is the same as the g-evaluation $\mathcal{E}_g[\xi|\mathcal{F}_t]$, which is a solution of the following BSDE

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = \xi + \int_t^T \kappa |Y_u| du - \int_t^T Y_u^1 dW_u - \int_t^T Y_u^2 dB_u,$$

where $\xi$ is a random variable such that $\xi \in \mathcal{F}_T$, $\mathbb{E}[|\xi|^2] < +\infty$.

It is reasonable that the bondholder chooses his convertible strategy to maximize his payoff, whereas, the issuer chooses his calling strategy to minimize his cost. So, the convertible bond pricing model can be formulated as Problem $\mathcal{R}_{tx}$, and the price of the convertible bond is equal to the value function $V_t(x)$. Where

$$G(t,x,r,y) = \rho + \kappa|y|, \quad V_t(x) = \rho e^x, \quad \nabla_t(x) = \max\{K, \rho e^x\}, \quad \varphi(x) = \max\{L, \rho e^x\}.$$  

Moreover, there exists a discounted rate process $c_u$ in this problem, which is equal to zero in $\mathcal{R}_{tx}$.

It is not difficult to check the problem satisfies Assumptions (D1), (D2) and (D3). Moreover, $\nabla = \nabla$ in the domain $\{x \geq \ln K - \ln \rho\}$, and Assumptions (V4) and (V5) hold true in the domain $\mathcal{D} \triangleq \{x < \ln K - \ln \rho\}$. If we make a slight modification in the previous proof, then we deduce from Theorem 4.9 that in the domain $\Omega \times [0,T] \times \mathcal{D}$, the price $V_t$ satisfies the following BSPDVI:

$$\begin{cases}
\quad dV_t = -\left( \mathcal{L}V_t + \kappa \sqrt{|\gamma_t \partial_x V_t|^2 + |Z_t|^2 + \rho} \right) dt + Z_t dB_t \quad \text{if } \rho e^x < V_t < K; \\
\quad dV_t \leq -\left( \mathcal{L}V_t + \kappa \sqrt{|\gamma_t \partial_x V_t|^2 + |Z_t|^2 + \rho} \right) dt + Z_t dB_t \quad \text{if } V_t = \rho e^x; \\
\quad dV_t \geq -\left( \mathcal{L}V_t + \kappa \sqrt{|\gamma_t \partial_x V_t|^2 + |Z_t|^2 + \rho} \right) dt + Z_t dB_t \quad \text{if } V_t = K; \\
\quad V_t(\ln K - \ln \rho) = K, \quad V_T(x) = \max\{L, \rho e^x\},
\end{cases}$$

where

$$\mathcal{L}V_t = \frac{\gamma_t^2}{2} \partial_{xx} V_t + \left( r_t - \frac{\gamma_t^2}{2} \right) \partial_x V_t - c_t V_t.$$
Moreover, the above BSPDVI has a strong solution. The optimal conversion strategy $\tau_1^*$ and the optimal calling strategy $\tau_2^*$ are given by
\[
\tau_1^* \triangleq \inf \{ s \in [t, T] : V_s(X_s) = \rho e^{X_s} \} \cup \inf \{ s \in [t, T] : X_s = \ln K - \ln \rho \} \wedge T,
\]
\[
\tau_2^* \triangleq \inf \{ s \in [t, T] : V_s(X_s) = K \} \wedge T.
\]

**Remark 6.1.** It is not difficult to show the discounted rate process, $c_t$, coincides with the coefficient, $c_t$, in the differential operator $\mathcal{L}$ if we make a slight modification in the proof of Theorem 3.2.

**Example 6.2.** *The optimal retirement and portfolio selection model in a finite horizon*

Suppose there are a riskless asset and $d_1$ risky assets in the market, where the risk-free rate is a positive constant $r$.

The price $P_0$ of the riskless asset and the price $P_t$ of the $i$-th risky asset are governed by the following SDE
\[
\left\{
\begin{aligned}
P_{0, s} &= P_0 + \int_t^s r P_{0, u} du, \\
P_{i, s} &= P_i + \int_t^s \alpha_i(u)P_{i, u} du + \sum_{j=1}^{d_1} \int_t^s \sigma_{ij}(u)P_{i, u} dW_t^j, 
\end{aligned}
\right.
\]
where $\alpha = (\alpha_1, \cdots, \alpha_{d_1})^\star$, $\hat{\Sigma} = (\hat{\sigma}^{ij})_{d_1 \times d_1}$ represents appreciation rate and volatility, respectively. They are deterministic continuous functions of time $u$ with values in $\mathbb{R}^{d_1}$ or $\mathbb{R}^{d_1 \times d_1}$. Moreover, $\hat{\Sigma}$ is non-degenerate, i.e., there exists a positive constant $\kappa$ such that $\xi^\star \hat{\Sigma}(u) \xi \geq \kappa \| \xi \|^2$ for any $\xi \in \mathbb{R}^{d_1}$, $u \in [0, T]$. And $W$ is a $d_1$-dimensional standard Brownian motion. In the following, we keep the previous notations.

Denoted by $\pi = (\pi_1, \cdots, \pi_{d_1})^\star$ and $c$, the amount of money invested in the risky assets and the consumption rate, respectively, which are $\mathcal{F}^W_t$-progressively measurable stochastic processes, and $c$ is nonnegative. Let $\tau$ be the time of retirement from labor, which belongs to $\mathcal{U}_{t, T}$.

Suppose that the agent can gain labor income at a nonnegative constant rate $\rho$ until retirement. So, the agent’s wealth process $Y$ is governed by
\[
Y_s^{(t, y)} = y + \int_t^s \left[ \pi_u^\star (\alpha(u) - r 1_{d_1}) + r Y_u^{(t, y)} - c_u + \rho I_{\{u \leq \tau\}} \right] du + \int_t^s \pi_u^\star \hat{\Sigma}(u) dW_u, \quad (6.1)
\]
where $1_{d_1}$ is the $d_1$-dimensional column vector of 1’s. We only consider the triple of control $(\tau, c, \pi)$ such that
\[
\int_t^T c_s + \| \pi_s \|^2 ds < \infty \text{ subject to } Y_s^{(t, y)} \geq \frac{\rho}{r} \left( e^{r(t-T)} - 1 \right), \forall s \in [t, T], \text{ and } Y_\tau^{(t, y)} \geq 0,
\]
which is called admissible policy. Denote by $\mathcal{A}(t, y)$ the set of all admissible policies. We assume the utility function $U(c)$ be strictly concave and satisfies
\[
U : (0, +\infty) \mapsto \mathbb{R}, \quad U'(0+) \triangleq \lim_{c \to 0^+} U'(c) = +\infty, \quad U'(+\infty) \triangleq \lim_{c \to \infty} U'(c) = 0.
\]
For example, $U$ is the constant relative risk aversion (CRRA), i.e.,

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad (0 < \gamma \neq 1) \quad \text{or} \quad U(c) = \ln c.$$

Our aim is to maximize the following utility

$$J(t, y; \tau, c, \pi) = \mathbb{E} \left[ \int_{t}^{\tau} e^{-\beta(u-t)} (U(c_u) - l) \, du + \int_{\tau}^{T} e^{-\beta(u-t)} U(c_u) \, du \right]$$

$$= \mathbb{E} \left\{ \int_{t}^{\tau} e^{-\beta(u-t)} (U(c_u) - l) \, du + \mathbb{E} \left[ \int_{\tau}^{T} e^{-\beta(u-t)} U(c_u) \, du \bigg| \mathcal{F}_{\tau}^W \right] \right\},$$

where $l$ is a constant disutility (or a utility loss) due to labor, and $(t, y) \in [0, T] \times (0, +\infty)$. Concretely speaking, we find an optimal strategy $(\tau^*, c^*, \pi^*) \in \mathcal{A}^1(t, y)$ such that,

$$J(t, y; \tau^*, c^*, \pi^*) = \hat{V}(t, y) \triangleq \sup \left\{ J(t, y; \tau, c, \pi) : (\tau, c, \pi) \in \mathcal{A}^1(t, y) \right\},$$

where

$$\mathcal{A}^1(t, y) \triangleq \left\{ (\tau, c, \pi) \in \mathcal{A} : \mathbb{E} \left[ \int_{t}^{\tau} e^{-\beta(u-t)} U^-(c_u) \, du + \int_{\tau}^{T} e^{-\beta(u-t)} U^-(c_u) \, du \right] < +\infty \right\}$$

with $U^- = \max\{0, -U\}$. It is clear that

$$\hat{V}(t, y) \leq \hat{V}^*(t, y) \triangleq \sup_{(\tau, c, \pi) \in \mathcal{A}^1(t, y)} \mathbb{E} \left\{ \int_{t}^{\tau} e^{-\beta(u-t)} (U(c_u) - l) \, du \right\}$$

$$+ e^{-\beta(\tau-t)} \sup_{(\bar{\tau}, \bar{c}, \bar{\pi}) \in \mathcal{A}^1(\tau, \gamma^{t,y})} \mathbb{E} \left[ \int_{\tau}^{T} e^{-\beta(u-\tau)} U(\bar{c}_u) \, du \bigg| \mathcal{F}_{\tau}^W \right] \right\},$$

$$= \sup_{(\tau, c, \pi) \in \mathcal{A}^1(t, y)} \mathbb{E} \left[ \int_{t}^{\tau} e^{-\beta(u-t)} (U(c_u) - l) \, du + e^{-\beta(\tau-t)} \hat{V}(\tau, Y^{t,y}_{\tau}) \right]$$

with

$$\hat{V}(t, y) = \sup_{(\tau, c, \pi) \in \mathcal{A}^1(t, y)} \mathbb{E} \left[ \int_{t}^{\tau} e^{-\beta(u-t)} U(c_u) \, du \right], \quad \mathcal{A}^1(t, y) \triangleq \{(\tau, c, \pi) \in \mathcal{A}^1(t, y) : \tau = t\}.$$

We have used the strong Markovian property of $Y$ in the last second equality.

On the other hand, for any $n \in \mathbb{N}_+$, there exist strategies $(\tau^n, c^n_1, \pi^n_1) \in \mathcal{A}^1(t, y)$ and $(\tau^n, c^n(\tau^n, Y^{t,y}_{\tau^n}), \pi^n(\tau^n, Y^{t,y}_{\tau^n})) \in \mathcal{A}^1_{\tau^n}(\tau^n, Y^{t,y}_{\tau^n})$ such that

$$\hat{V}^*(t, y) \leq \mathbb{E} \left[ \int_{t}^{\tau^n} e^{-\beta(u-t)} \left( U(c^n_{1,u}) - l \right) \, du + e^{-\beta(\tau^n-t)} \hat{V}(\tau^n, Y^{t,y}_{\tau^n}) \right] + \frac{1}{n},$$

$$\hat{V}(\tau^n, Y^{t,y}_{\tau^n}) \leq \mathbb{E} \left[ \int_{\tau^n}^{T} e^{-\beta(u-\tau^n)} U(c^n_u(\tau^n, Y^{t,y}_{\tau^n})) \, du \bigg| \mathcal{F}_{\tau^n}^W \right] + \frac{1}{n},$$

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Define
\[ \tilde{\tau}^n = \tau^n, \quad \tilde{c}^n = c^n I_{(t \leq \tau^n)} + c^n(\tau^n, Y^{t,y}_{\tau^n}) I_{(t > \tau^n)}, \quad \tilde{\pi}^n = \pi^n I_{(t \leq \tau^n)} + \pi^n(\tau^n, Y^{t,y}_{\tau^n}) I_{(t > \tau^n)}. \]

We can check that \((\tilde{\tau}^n, \tilde{c}^n, \tilde{\pi}^n) \in \mathcal{A}^1(t, y)\) and
\begin{equation}
J(t, y; \tilde{\tau}^n, \tilde{c}^n, \tilde{\pi}^n) \geq \tilde{V}^*(t, y) + \frac{2}{n}.
\end{equation}

Hence, we have
\begin{equation}
\tilde{V}(t, y) = \sup_{(\tau, c, \pi) \in \mathcal{A}(t, y)} \mathbb{E} \left[ \int_t^T e^{-\beta(u-t)} \left( U(c_u) - l \right) du + e^{-\beta(\tau-t)} \tilde{V}(\tau, Y^{t,y}_\tau) \right]
\end{equation}
subject to (6.1), where \(\tilde{V}\) is the value function of the following stochastic control problem
\begin{equation}
\tilde{V}(t, y) = \sup_{(\tau, c, \pi) \in \mathcal{A}_1^1(t, y)} \mathbb{E} \left[ \int_t^T e^{-\beta(u-t)} U(c_u) du \right]
\end{equation}
subject to (6.1).

Following the idea in \([18, 19]\), we use the martingale and dual method to transform the original problem into a standard stopping time problem.

Define the discount process \(D(s)\), the relative risk \(R(s)\), the exponential martingale \(M_s\) and the state-price-density process \(S_s\) as \(D(s) = e^{-r(s-t)}\), and
\[ R(s) = (\tilde{\Sigma}(s))^{-1} \left( \alpha(s) - r 1_{d_1} \right), \quad M_s = \exp \left\{ - \int_t^s R^*(u) dW_u - \frac{1}{2} \int_t^s \|R(u)\|^2 du \right\}, \]
and \(S_s = D(s)M_s\).

Applying the Itô formula, we have
\[ dS_u \left[ Y_u - \frac{\theta}{r} \left( e^{ru-rT} - 1 \right) I_{[u \leq \tau]} \right] = -S_u c_u du + \pi_u \tilde{\Sigma}(u) - \left[ Y_u - \frac{\theta}{r} \left( e^{ru-rT} - 1 \right) I_{[u \leq \tau]} \right] R^*(u) dW_u. \]

So, we have
\begin{equation}
S_s \left[ Y_s - \frac{\theta}{r} \left( e^{rs-rT} - 1 \right) \right] + \int_t^s S_u c_u du = y - \frac{\theta}{r} \left( e^{rt-rT} - 1 \right)
\end{equation}
+ local martingale if \( t \leq s \leq \tau \),
\begin{equation}
S_s Y_s + \int_t^s S_u c_u du = y + \text{local martingale if } s \geq t \geq \tau.
\end{equation}

We have used the fact \(S_t = 1\) in the first equality. Since the left-hand sides of the equalities are nonnegative and the right-hand sides are local martingales. Applying the optimal sampling theorem, we have
\begin{equation}
\mathbb{E} \left\{ S_\tau \left[ Y_\tau - \frac{\theta}{r} \left( e^{rt-rT} - 1 \right) \right] + \int_t^\tau S_u c_u du \right\} \leq y - \frac{\theta}{r} \left( e^{rt-rT} - 1 \right) \text{ if } t \leq \tau; \quad (6.4)
\end{equation}
\begin{equation}
\mathbb{E} \left[ S_T Y_T + \int_t^T S_u c_u du \right] \leq y \text{ if } \tau \leq t \leq T.
\end{equation}
Remarking 6.2. According to [18, 19], the equalities hold in (6.4) and (6.5) in some cases, and they are equivalent to (6.1) in some senses.

First, we analyze the property of $\hat{V}$. For a Lagrange multiplier $z \geq 0$, we transform the stochastic control problem (6.3) subject to (6.5) into

$$
\sup_{(\tau, c, \pi) \in A_1(t, u)} \mathbb{E} \left[ \int_t^T \left[ e^{-\alpha(u-t)} U(c) - z S_u c_u \right] du + (0 - z S_T Y_T) \right].
$$

According to [18, Theorem 6.11], the above problem can be described as

$$
\tilde{V}_1(t, z) \triangleq \sup_{y \geq 0} \left( \hat{V}(t, y) - z y \right) = \mathbb{E} \left[ \int_t^T e^{-\alpha(u-t)} \tilde{U}_1(K_u) du \right]
$$

with $K_u = z e^{\alpha(u-t)} S_u$, where $\tilde{U}_1$ is the convex dual functions of the concave function $U$, i.e.,

$$
\tilde{U}_1(z) = \sup_{y \geq 0} \left( U(y) - z y \right), \quad \forall \ z \geq 0.
$$

Moreover, we have used the fact that the convex dual function of 0 is 0, and $\hat{V}$ is concave and nondecreasing.

As in Example 6.1, we apply the logarithmic transformation to $\hat{V}$ and $K_u$, and define

$$
X_u = \ln K_u, \quad x = \ln z, \quad G_1(x) = \tilde{U}_1(e^x), \quad \tilde{V}_1(t, x) = \tilde{V}_1(t, e^x).
$$

It is not difficult to check that $X_s$ is governed by

$$
X_s = x + \int_t^s \left( \beta - r - \frac{|R(u)|^2}{2} \right) du - \int_t^s R^*(u) dW_u,
$$

and

$$
\tilde{V}_1(t, x) = \mathbb{E} \left[ \int_t^T e^{-\alpha(u-t)} G_1(X_u) du \right].
$$

In view of Feynman-Kac formula, $\tilde{V}_1$ satisfies the following PDE:

$$
\begin{cases}
\partial_t \tilde{V}_1 + \mathcal{L} \tilde{V}_1 + G_1 = 0; \\
\tilde{V}_1(T, x) = 0,
\end{cases}
$$

where

$$
\mathcal{L} V = \frac{|R|^2}{2} \partial_{xx} V + \left( \beta - r - \frac{|R|^2}{2} \right) \partial_x V - \beta V.
$$

From the regularity of PDE and the comparison principle, it is not difficult to deduce that $\tilde{V}_1 \in H^{2,2}_2$, $\partial_t \tilde{V}_1 \in H^{0,2}_2$. Moreover, we can prove that $\tilde{V}_1 \in W^{2,1}_{p,\text{loc}}$ for any fixed $p > 1$, which means that $\tilde{V}_1, \partial_t \tilde{V}_1, \partial_x \tilde{V}_1, \partial_{xx} \tilde{V}_1 \in L^p([0, T] \times [-m, m])$ for any $m \in \mathbb{N}_+$.

Next, we use the similar method in [19] to transform problem (6.2) subject to (6.4) into a standard stopping time problem.

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Define the convex dual functions $\tilde{U}, \tilde{V}$ of the concave function $U, \hat{V}$ as

$$
\tilde{U}(t, z) = \sup_{y \geq 0} \{ U(y) - l - yz \} = \sup_{y \geq 0} [U(y) - yz] - l,
$$

$$
\tilde{V}(t, z) = \sup_{y \geq 0} \left[ \tilde{V}(t, y) - \left[ y - \frac{\theta}{r} \left( e^{rT} - 1 \right) \right] z \right] = \tilde{V}_1(t, z) + \frac{\theta}{r} \left( e^{rT} - 1 \right) z
$$

for any $z \geq 0$.

Moreover, if we let $I_1, I_2(t, \cdot)$ be the inverse function of $U', \partial_y \tilde{V}(t, \cdot)$, respectively. Then

$$
\tilde{U}(t, z) = U(I_1(z)) - zI_1(z) - l,
$$

$$
\tilde{V}(t, z) = \tilde{V}(t, I_2(z)) - zI_2(z) + \frac{\theta}{r} \left( e^{rT} - 1 \right) z.
$$

According to (6.2), for any $z \geq 0$, we have

$$
J^*(y, t; \tau, c, \pi)
$$

$$
\triangleq \int_t^\tau e^{-\beta(u-t)} \left( U(c_u) - l \right) du + e^{-\beta(\tau-t)}\tilde{V}(\tau, Y^t_u, y)
$$

$$
\leq E \left[ \int_t^\tau e^{-\beta(u-t)} \tilde{U} \left( u, z e^{\beta(u-t)} S_u \right) du + e^{-\beta(\tau-t)}\tilde{V} \left( \tau, z e^{\beta(\tau-t)} S_\tau \right) \right]
$$

$$
+ z E \left\{ \int_t^\tau S_u c_u du + S_\tau \left[ Y_\tau - \frac{\theta}{r} \left( e^{rT} - 1 \right) \right] \right\}
$$

$$
\leq E \left[ \int_t^\tau e^{-\beta(u-t)} \tilde{U} \left( z e^{\beta(u-t)} S_u \right) du + e^{-\beta(\tau-t)}\tilde{V} \left( \tau, z e^{\beta(\tau-t)} S_\tau \right) \right]
$$

$$
+ z \left[ y - \frac{\theta}{r} \left( e^{rT} - 1 \right) \right].
$$

Define

$$
\tilde{V}(t, z) = \sup_{\tau \in [t, T]} \tilde{J}(t, z; \tau), \quad \tilde{J}(t, z; \tau) \triangleq E \left[ \int_t^\tau e^{-\beta(u-t)} \tilde{U} \left( K_u \right) du + e^{-\beta(\tau-t)}\tilde{V} \left( \tau, K_\tau \right) \right],
$$

where $K_u$ is same as the above. Similarly as in [19], we have

$$
\tilde{V}(t, y) \leq \inf_{z \geq 0} \left\{ \tilde{V}(t, z) + z \left[ y - \frac{\theta}{r} \left( e^{rT} - 1 \right) \right] \right\},
$$

with the equality holding if $\tilde{V}(t, z)$ is differentiable with respect to $z$.

So, it is sufficient to consider the optimal problem of $\tilde{V}(t, z)$. As the above, we apply the following transformation and let

$$
X_u = \ln K_u, \quad x = \ln z, \quad G(x) = \tilde{U}(e^x), \quad \tilde{V}(t, x) = \tilde{V}(t, e^x), \quad V(t, x) = \tilde{V}(t, e^x),
$$

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then $V(t, x)$ is the value function of the following standard optimal stopping time problem, where the state equation is (6.6), and the payoff function is

$$P(t, x; \tau) \triangleq \mathbb{E} \left[ \int_t^\tau e^{-\beta(u-t)} G(X_u^{t,x}) du + e^{-\beta(\tau-t)} V(\tau, X_\tau^{t,x}) \right].$$

We find a optimal stopping time $\tau^*$ such that

$$P(t, x; \tau^*) = V(t, x) = \sup\{P(t, x; \tau) : \tau \in \mathcal{U}_{t,T}\}.$$

In view of Theorem 5.3, $V$ satisfies the following PDVI:

$$
\begin{cases}
    dV_t(x) = -\left( L V_t(x) + G(x) \right) dt & \text{if } V > V; \\
    dV_t(x) \leq -\left( L V_t(x) + G(x) \right) dt & \text{if } V = V; \\
    V_T = V_T,
\end{cases}
$$

where $L$ is defined in (6.7). Moreover, the above BSPDVI has a strong solution $V \in H_2^{2,2}$. Since, it is a deterministic PDVI, then $V \in W_2^{2,1}$. So, the above PDVI is written into the following:

$$\partial_t V + LV + G = \chi_{\{ V = V \}} \left( \partial_t V + LV \right) \text{ a.e. in } [0, T] \times \mathbb{R}.$$ 

According to the regularity theorem of PDE, the solution $V \in W_2^{2,1}$ for any $p > 1$. Then the Sobolev embedding theorem implies that $\partial_x V$ is continuous, and $\tilde{V}(t, z)$ is differentiable in $z$. Hence, the value function $\tilde{V}$ of the original problem is given by

$$\tilde{V}(t, y) = \inf_{z \geq 0} \left\{ \tilde{V}(t, z) + z \left[ y - \frac{\theta}{r} (e^{rt} - e^{rT}) \right] \right\}.$$

References


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